

Riesz Transform Characterizations of Hardy Spaces Associated to Degenerate Elliptic Operators

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Abstract Let w be a Muckenhoupt $A_2(\mathbb{R}^n)$ weight and $L_w := -w^{-1} \operatorname{div}(A \nabla)$ the degenerate elliptic operator on the Euclidean space \mathbb{R}^n . In this article, the authors establish the Riesz transform characterization of the Hardy space $H_{L_w}^p(\mathbb{R}^n)$ associated with L_w , for $w \in A_q(\mathbb{R}^n)$ and $w^{-1} \in A_{2-\frac{2}{q}}(\mathbb{R}^n)$ with $n \geq 3$, $q \in [1, 2]$ and $p \in (q(\frac{1}{r} + \frac{q-1}{2} + \frac{1}{n})^{-1}, 1]$ if, for some $r \in [1, 2)$, $\{tL_w e^{-tL_w}\}_{t \geq 0}$ satisfies the weighted $L^r - L^2$ full off-diagonal estimate.

1 Introduction

The theory of classical real Hardy spaces $H^p(\mathbb{R}^n)$ originates from Stein and Weiss [39] in the early 1960s. Since then, this real-variable theory received continuous development and now is increasingly mature; see, for example, [24, 38]. It is well known that the Hardy space $H^p(\mathbb{R}^n)$ is a suitable substitute of the Lebesgue space $L^p(\mathbb{R}^n)$, when $p \in (0, 1]$, and plays important roles in various fields of analysis and partial differential equations. Notice that $H^p(\mathbb{R}^n)$ is essentially associated with the Laplace operator $\Delta := \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$; see [22, 27] for instance.

The motivation to study the Hardy spaces associated with different operators (for example, the second order elliptic operator $-\operatorname{div}(A \nabla)$ and the Schrödinger operator $-\Delta + V$) comes from characterizing the boundedness of the associated Riesz transforms and the regularity of solutions of the associated equations; see, for example, [3, 21, 22, 2, 42, 29, 26, 27, 20, 19, 6].

Consider now a degenerate elliptic operator. Let w be a Muckenhoupt $A_2(\mathbb{R}^n)$ weight and $A(x) := (A_{ij}(x))_{i,j=1}^n$ be a matrix of complex-valued, measurable functions on \mathbb{R}^n satisfying the *degenerate elliptic condition* that there exist positive constants $\lambda \leq \Lambda$ such that, for almost every $x \in \mathbb{R}^n$ and all $\xi, \eta \in \mathbb{C}^n$,

$$(1.1) \quad |\langle A(x)\xi, \eta \rangle| \leq \Lambda w(x)|\xi||\eta|$$

and

$$(1.2) \quad \Re \langle A(x)\xi, \xi \rangle \geq \lambda w(x)|\xi|^2,$$

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here and hereafter, $\Re z$ denotes the *real part* of z for any $z \in \mathbb{C}$. The associated *degenerate elliptic operator* L_w is defined by setting, for all $f \in D(L_w) \subset \mathcal{H}_0^1(w, \mathbb{R}^n)$,

$$(1.3) \quad L_w f := -\frac{1}{w} \operatorname{div}(A \nabla f),$$

which is interpreted in the usual weak sense via the sesquilinear form, where $D(L_w)$ denotes the domain of L_w . Here and in what follows, $\mathcal{H}_0^1(w, \mathbb{R}^n)$ denotes the *weighted Sobolev space*, which is defined to be the closure of $C_c^\infty(\mathbb{R}^n)$ with respect to the *norm*

$$\|f\|_{\mathcal{H}_0^1(w, \mathbb{R}^n)} := \left\{ \int_{\mathbb{R}^n} [|f(x)|^2 + |\nabla f(x)|^2] w(x) dx \right\}^{1/2}.$$

The *sesquilinear form* \mathfrak{a} associated with L_w is defined by setting, for all $f, g \in \mathcal{H}_0^1(w, \mathbb{R}^n)$,

$$\mathfrak{a}(f, g) := \int_{\mathbb{R}^n} [A(x) \nabla f(x)] \cdot \overline{\nabla g(x)} dx.$$

In the case $w \equiv 1$, the degenerate elliptic operator L_w reduces to the usual second order elliptic operator $L = -\operatorname{div}(A \nabla)$. Therefore, L_w may be considered as a generalization of the usual uniformly elliptic operator.

Operators of the form (1.3) and the associated elliptic equations were first considered by Fabes, Kenig and Serapioni [23] and have been considered by a number of other authors (see, for example, [8, 9, 7, 28] and, especially, some recent articles by Cruz-Uribe and Rios [13, 14, 15, 16]). We point out that, when w is a weight in the Muckenhoupt class $A_2(\mathbb{R}^n)$, the space $\mathcal{H}_0^1(w, \mathbb{R}^n)$ was first studied by Fabes et al. in [23], where the local weighted Sobolev embedding theorem and the Poincaré inequality were proved to hold true.

Let L_w be a degenerate elliptic operator as in (1.3) with w in the Muckenhoupt class of $A_2(\mathbb{R}^n)$ weights (see Subsection 2.1 below for their exact definitions). The main purpose of this article is to establish the Riesz transform characterizations of Hardy spaces $H_{L_w}^p(\mathbb{R}^n)$ associated with L_w (see Theorem 1.4 below).

This article may be viewed in part as a sequel to [43], where the non-tangential maximal function characterizations of Hardy spaces $H_{L_w}^p(\mathbb{R}^n)$ associated with L_w and the boundedness of the associated Riesz transform on these spaces have been studied.

To state the main results of this article, we first introduce some definitions and notation. Let $w \in A_2(\mathbb{R}^n)$, L_w be as in (1.3) and $f \in L^2(w, \mathbb{R}^n)$, where $L^2(w, \mathbb{R}^n)$ denotes the *weighted Lebesgue space* with the *norm*

$$\|f\|_{L^2(w, \mathbb{R}^n)} := \left\{ \int_{\mathbb{R}^n} |f(x)|^2 w(x) dx \right\}^{\frac{1}{2}}.$$

It is well known that, if $w \in A_2(\mathbb{R}^n)$, $L^2(w, \mathbb{R}^n)$ is a space of homogenous type in the sense of Coifman and Weiss [11, 12], since $w(x) dx$ is a doubling measure. In what follows, let $\mathbb{R}_+^{n+1} := \mathbb{R}^n \times (0, \infty)$. For any $f \in L^2(w, \mathbb{R}^n)$ and $x \in \mathbb{R}^n$, the *square function* $\mathcal{S}_{L_w}(f)$ associated with L_w is defined by setting

$$\mathcal{S}_{L_w}(f)(x) := \left[\iint_{\Gamma(x)} \left| t^2 L_w e^{-t^2 L_w}(f)(y) \right|^2 w(y) \frac{dy}{w(B(x, t))} \frac{dt}{t} \right]^{1/2},$$

where $B(x, t) := \{y \in \mathbb{R}^n : |x - y| < t\}$, $w(B(x, t)) := \int_{B(x, t)} w(y) dy$ and

$$(1.4) \quad \Gamma_\alpha(x) := \{(y, t) \in \mathbb{R}_+^{n+1} : |x - y| < \alpha t\}$$

denotes the *cone of aperture α with vertex x* . In particular, if $\alpha = 1$, we write $\Gamma(x)$ instead of $\Gamma_\alpha(x)$.

The Hardy spaces $H_{L_w}^p(\mathbb{R}^n)$ associated with L_w were defined in [43, Definition 1.1] as follows.

Definition 1.1 ([43]). Let $p \in (0, 1]$, $w \in A_2(\mathbb{R}^n)$ and L_w be the degenerate elliptic operator as in (1.3) with the matrix A satisfying the degenerate elliptic conditions (1.1) and (1.2). The *Hardy space* $H_{L_w}^p(\mathbb{R}^n)$, associated with L_w , is defined as the completion of the space

$$\{f \in L^2(w, \mathbb{R}^n) : \|S_{L_w}(f)\|_{L^p(w, \mathbb{R}^n)} < \infty\}$$

with respect to the *quasi-norm*

$$\|f\|_{H_{L_w}^p(\mathbb{R}^n)} := \|S_{L_w}(f)\|_{L^p(w, \mathbb{R}^n)}.$$

We introduce the following Hardy spaces associated with the Riesz transform, which, when $w \equiv 1$, is just the one defined in [27, p. 7].

Definition 1.2. Let $p \in (0, 1]$, $w \in A_2(\mathbb{R}^n)$ and L_w be the degenerate elliptic operator as in (1.3) with the matrix A satisfying the degenerate elliptic conditions (1.1) and (1.2). The *Hardy space* $H_{L_w, \text{Riesz}}^p(\mathbb{R}^n)$ is defined as the completion of the space

$$\{f \in L^2(w, \mathbb{R}^n) : \nabla L_w^{-1/2} f \in H_w^p(\mathbb{R}^n)\}$$

with respect to the *quasi-norm*

$$\|f\|_{H_{L_w, \text{Riesz}}^p(\mathbb{R}^n)} := \|\nabla L_w^{-1/2} f\|_{H_w^p(\mathbb{R}^n)}.$$

Before establishing the Riesz transform characterization of $H_{L_w}^p(\mathbb{R}^n)$, we first introduce the following definition of weighted full off-diagonal estimates, which is a generalization of full off-diagonal estimates in spaces of homogeneous type (see [1, Definition 3.1]).

Definition 1.3. Let $w \in A_\infty(\mathbb{R}^n)$ and $1 \leq p \leq q < \infty$. A family $\{T_t\}_{t \geq 0}$ of sublinear operators is said to satisfy the *weighted $L^p - L^q$ full off-diagonal estimates*, denoted by $T_t \in \mathcal{F}_w(L^p - L^q)$, if there exist positive constants $C, c \in (0, \infty)$ such that, for any closed sets E, F of \mathbb{R}^n and $f \in L^p(w^{\frac{p}{2}}, E)$ with $\text{supp } f \subset E$,

$$\left\{ \int_F |T_t(f)(x)|^q [w(x)]^{\frac{q}{2}} dx \right\}^{\frac{1}{q}} \leq C t^{-\frac{n}{2}(\frac{1}{p} - \frac{1}{q})} e^{-c \frac{[d(E, F)]^2}{t}} \left\{ \int_E |f(x)|^p [w(x)]^{\frac{p}{2}} dx \right\}^{\frac{1}{p}}.$$

The following theorem establishes the Riesz transform characterizations of $H_{L_w}^p(\mathbb{R}^n)$.

Theorem 1.4. *Let $q \in [1, 2]$ and $w \in A_q(\mathbb{R}^n)$ satisfy $w^{-1} \in A_{2-\frac{2}{n}}(\mathbb{R}^n)$ with $n \geq 3$. Assume $tL_w e^{-tL_w} \in \mathcal{F}_w(L^r - L^2)$ for some $r \in [1, 2)$. Then, for $p \in (q(\frac{1}{r} + \frac{q-1}{2} + \frac{1}{n})^{-1}, 1]$, the Hardy spaces $H_{L_w, \text{Riesz}}^p(\mathbb{R}^n)$ and $H_{L_w}^p(\mathbb{R}^n)$ coincide with equivalent quasi-norms.*

Remark 1.5. (i) Since we need to apply the weighted Sobolev inequality (see (2.17) below) in the proof of Theorem 1.4, to this end, we need to assume $w^{-1} \in A_{2-\frac{2}{n}}(\mathbb{R}^n)$ with $n \geq 3$ in Theorem 1.4.

(ii) In the case $w \equiv 1$, Theorem 1.4 reduces to [27, Proposition 5.18], where Hofmann et al. first established the Riesz transform characterizations of Hardy spaces $H_L^p(\mathbb{R}^n)$ associated with the second order elliptic operators $L := -\text{div}(A\nabla)$; we point out that, in this case, the range of p of Theorem 1.4 coincides with that of [27, Proposition 5.18].

From Theorem 1.4 and Remark 2.5 below, we immediately deduce the following conclusions, the details being omitted.

Corollary 1.6. *Let $q \in [1, 2]$, $s \in (1, \infty]$ and $w \in A_q(\mathbb{R}^n) \cap RH_s(\mathbb{R}^n)$ satisfy $w^{-1} \in A_{2-\frac{2}{n}}(\mathbb{R}^n)$ with $n \geq 3$. If the matrix A associated with L_w is real symmetric, then, for all $p \in (q[\frac{1}{2}(1 - \frac{1}{s}) + \frac{q}{2} + \frac{1}{n}]^{-1}, 1]$, the Hardy spaces $H_{L_w, \text{Riesz}}^p(\mathbb{R}^n)$ and $H_{L_w}^p(\mathbb{R}^n)$ coincide with equivalent quasi-norms.*

By Theorem 1.4 and Proposition 2.6 below, we immediately conclude the following conclusion, the details being omitted.

Corollary 1.7. *Let $q \in [1, \frac{4}{3})$ and $w \in A_q(\mathbb{R}^3)$ satisfy $w^{-1} \in A_{\frac{4}{3}}(\mathbb{R}^3)$. Then, for $p \in (\frac{6q}{4+3q}, 1]$, the Hardy spaces $H_{L_w, \text{Riesz}}^p(\mathbb{R}^3)$ and $H_{L_w}^p(\mathbb{R}^3)$ coincide with equivalent quasi-norms.*

We prove Theorem 1.4 by following the strategy used in [27, Proposition 5.18]. The proof of Theorem 1.4 rests on the atomic decomposition of the weighted Hardy-Sobolev spaces (see Theorem 1.8 below).

In what follows, let $\mathcal{S}(\mathbb{R}^n)$ denote the space of all Schwartz functions and $\mathcal{S}'(\mathbb{R}^n)$ be the space of all Schwartz distributions.

Let $\psi \in \mathcal{S}(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} \psi(x) dx = 1$ and $\psi_t(x) := t^{-n} \psi(\frac{x}{t})$ for all $x \in \mathbb{R}^n$ and $t \in (0, \infty)$. For all $f \in \mathcal{S}'(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, the non-tangential maximal function $\psi_\nabla^*(f)(x)$ is defined by setting

$$\psi_\nabla^*(f)(x) := \sup_{|x-y| < t, t \in (0, \infty)} |(\psi_t * f)(y)|.$$

Then, for $p \in (0, 1]$ and $w \in A_\infty(\mathbb{R}^n)$, $f \in \mathcal{S}'(\mathbb{R}^n)$ is said to belong to the weighted Hardy space $H_w^p(\mathbb{R}^n)$, if $\psi_\nabla^*(f) \in L^p(w, \mathbb{R}^n)$; moreover, its norm is given by $\|f\|_{H_w^p(\mathbb{R}^n)} := \|\psi_\nabla^*(f)\|_{L^p(w, \mathbb{R}^n)}$.

Let $\mathcal{S}_0(\mathbb{R}^n)$ be the space of all Schwartz functions φ that satisfy $\int_{\mathbb{R}^n} \varphi(x) dx = 0$. Then $\mathcal{S}_0(\mathbb{R}^n)$ is a subspace of $\mathcal{S}(\mathbb{R}^n)$ that inherits the same topology as $\mathcal{S}(\mathbb{R}^n)$. We denote the dual of $\mathcal{S}_0(\mathbb{R}^n)$ by $\mathcal{S}'_0(\mathbb{R}^n)$.

Let $p \in (0, 1]$ and $w \in A_\infty(\mathbb{R}^n)$. The weighted Hardy-Sobolev space is defined as the set

$$H_w^{1,p}(\mathbb{R}^n) := \{f \in \mathcal{S}'_0(\mathbb{R}^n) : \nabla f \in H_w^p(\mathbb{R}^n)\}$$

with the quasi-norm

$$\|f\|_{H_w^{1,p}(\mathbb{R}^n)} := \|\nabla f\|_{H_w^p(\mathbb{R}^n)} := \sum_{j=1}^n \|\partial_j f\|_{H_w^p(\mathbb{R}^n)},$$

where $\nabla f := (\partial_1 f, \dots, \partial_n f)$ stands for the distributional derivatives of f and $\nabla f \in H_w^p(\mathbb{R}^n)$ means that, for all $j \in \{1, \dots, n\}$, $\partial_j f \in H_w^p(\mathbb{R}^n)$.

In what follows, for a subset $E \subset \mathbb{R}^n$, $C_c^\infty(E)$ denotes the set of all C^∞ functions with compact support in E . For a ball B of \mathbb{R}^n and $w \in A_\infty(\mathbb{R}^n)$, we define $H_0^1(w, B)$ to be the closure of $C_c^\infty(B)$ with respect to the norm

$$\|f\|_{H_0^1(w, B)} := \left\{ \int_B [|f(x)|^2 + |\nabla f(x)|^2] w(x) dx \right\}^{\frac{1}{2}}.$$

Let $p \in (0, 1]$, $w \in A_\infty(\mathbb{R}^n)$ and $B \subset \mathbb{R}^n$ be a ball. A function $a \in H_0^1(w, B)$ is called an $H_w^{1,p}(\mathbb{R}^n)$ -atom if

- (i) $\text{supp } a \subset B$;
- (ii) $\|a\|_{L^2(w, B)} \leq r_B \|\nabla a\|_{L^2(w, B)}$, where r_B denotes the radius of B ;
- (iii) $\|\nabla a\|_{L^2(w, B)} \leq [w(B)]^{\frac{1}{2} - \frac{1}{p}}$.

The following theorem gives an atomic decomposition for distributions in $H_w^{1,p}(\mathbb{R}^n)$, which plays a key role in the proof of Theorem 1.4.

Theorem 1.8. *Let $w \in A_2(\mathbb{R}^n)$, $p \in (0, 1]$ and $f \in \mathcal{H}_0^1(w, \mathbb{R}^n) \cap H_w^{1,p}(\mathbb{R}^n)$. Then there exist a sequence of $H_w^{1,p}(\mathbb{R}^n)$ -atoms, $\{\beta_k\}_{k \in \mathbb{N}}$, and a sequence of numbers, $\{\lambda_k\}_{k \in \mathbb{N}} \subset \mathbb{C}$, such that*

$$(1.5) \quad f = \sum_{k=1}^{\infty} \lambda_k \beta_k \quad \text{in } \mathcal{S}'_0(\mathbb{R}^n),$$

and

$$\nabla f = \sum_{k=1}^{\infty} \lambda_k \nabla \beta_k \quad \text{in } L^2(w, \mathbb{R}^n).$$

Moreover, there exists a positive constant C , independent of f , such that

$$\left\{ \sum_{k=1}^{\infty} |\lambda_k|^p \right\}^{\frac{1}{p}} \leq C \|\nabla f\|_{H_w^p(\mathbb{R}^n)}.$$

Recall that Lou and Yang in [33] gave an atomic characterization for the classical Hardy-Sobolev space $H^{1,1}(\mathbb{R}^n)$. Following their methods therein, we prove Theorem 1.8 through the atomic decomposition for tent spaces, which was originally introduced in [10]. However, we point out that the proof of Theorem 1.8 is slightly different from that of [33, Lemma 1]. We prove the size condition of the $H_w^{1,p}(\mathbb{R}^n)$ -atoms by the local weighted Sobolev embedding theorems in [23], for $A_2(\mathbb{R}^n)$ weights, instead of [36, Chapter 3, Theorem 3.3.3] which was used in the corresponding proof of [33, Lemma 1].

This article is organized as follows. In Subsection 2.1, we first recall some notions and results on Muckenhoupt weights; then, in Subsection 2.2, we establish the weighted off-diagonal estimates for L_w ; in Subsection 2.3, we introduce the weighted tent space and establish its atomic decomposition. Section 3 is devoted to the proof of Theorem 1.8, while Theorem 1.4 is proved in Section 4.

We end this section by making some conventions on notation. Throughout this article, L_w always denotes a degenerate elliptic operator as in (1.3). We denote by C a positive constant which is independent of the main parameters, but it may vary from line to line. We also use $C_{(\alpha, \beta, \dots)}$ to denote a positive constant depending on the parameters α, β, \dots . The symbol $f \lesssim g$ means that $f \leq Cg$. If $f \lesssim g$ and $g \lesssim f$, then we write $f \sim g$. For any measurable subset E of \mathbb{R}^n , we denote by E^c the set $\mathbb{R}^n \setminus E$. Let $\mathbb{N} := \{1, 2, \dots\}$ and $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. For any closed set $F \subset \mathbb{R}^n$, we let

$$(1.6) \quad R(F) := \bigcup_{x \in F} \Gamma(x),$$

where $\Gamma(x)$ for all $x \in F$ is as in (1.4) with $\alpha = 1$. For any ball $B := (x_B, r_B) \subset \mathbb{R}^n$, $\alpha \in (0, \infty)$ and $j \in \mathbb{N}$, we let $\alpha B := B(x_B, \alpha r_B)$,

$$(1.7) \quad U_0(B) := B \quad \text{and} \quad U_j(B) := (2^j B) \setminus (2^{j-1} B).$$

2 Preliminaries

In this section, we first recall the definition of the *Muckenhoupt weights* and some of their properties. Then we establish the weighted full off-diagonal estimates for L_w , which play a key role in the proofs of our main results. Finally, we recall the definition of the weighted tent space and its atomic decomposition, which is used in Section 3.

2.1 Muckenhoupt weights

Let $q \in [1, \infty)$. A nonnegative and locally integrable function w on \mathbb{R}^n is said to belong to the *Muckenhoupt class* $A_q(\mathbb{R}^n)$, if there exists a positive constant C such that, for all balls $B \subset \mathbb{R}^n$, when $q \in (1, \infty)$,

$$\frac{1}{|B|} \int_B w(x) dx \left\{ \frac{1}{|B|} \int_B [w(x)]^{-\frac{1}{q-1}} dx \right\}^{q-1} \leq C$$

or, when $q = 1$,

$$\frac{1}{|B|} \int_B w(x) dx \leq C \operatorname{ess\,inf}_{x \in B} w(x).$$

We also let $A_\infty(\mathbb{R}^n) := \cup_{q \in [1, \infty)} A_q(\mathbb{R}^n)$ and $w(E) := \int_E w(x) dx$ for any measurable set $E \subset \mathbb{R}^n$.

Let $r \in (1, \infty]$. A nonnegative locally integrable function w is said to belong to the *reverse Hölder class* $RH_r(\mathbb{R}^n)$, if there exists a positive constant C such that, for all balls $B \subset \mathbb{R}^n$,

$$\left\{ \frac{1}{|B|} \int_B [w(x)]^r dx \right\}^{1/r} \leq C \frac{1}{|B|} \int_B w(x) dx,$$

where we replace $\{\frac{1}{|B|} \int_B [w(x)]^r dx\}^{1/r}$ by $\|w\|_{L^\infty(B)}$ when $r = \infty$.

We recall some properties of Muckenhoupt weights and reverse Hölder classes in the following two lemmas (see, for example, [18] for their proofs).

Lemma 2.1. (i) *If $1 \leq p \leq q \leq \infty$, then $A_1(\mathbb{R}^n) \subset A_p(\mathbb{R}^n) \subset A_q(\mathbb{R}^n)$.*
(ii) $A_\infty(\mathbb{R}^n) := \cup_{p \in [1, \infty)} A_p(\mathbb{R}^n) = \cup_{r \in (1, \infty]} RH_r(\mathbb{R}^n)$.

Lemma 2.2. *Let $q \in [1, \infty)$ and $r \in (1, \infty]$. If a nonnegative measurable function $w \in A_q(\mathbb{R}^n) \cap RH_r(\mathbb{R}^n)$, then there exists a constant $C \in (1, \infty)$ such that, for all balls $B \subset \mathbb{R}^n$ and any measurable subset E of B ,*

$$C^{-1} \left(\frac{|E|}{|B|} \right)^q \leq \frac{w(E)}{w(B)} \leq C \left(\frac{|E|}{|B|} \right)^{\frac{r-1}{r}}.$$

2.2 Weighted full off-diagonal estimates for L_w

In this subsection, we first recall the definition of weighted off-diagonal estimates on balls from [1]. Then we show that, if the matrix A associated with L_w is real symmetric, then, for any $p \in [1, 2)$, $tL_w e^{-tL_w} \in \mathcal{F}_w(L^p - L^2)$. Finally, we prove that, in the general case, for $n \geq 3$, $k \in \mathbb{Z}_+$ and $p_- = \frac{2n}{n+2}$, $(tL_w)^k e^{-tL_w} \in \mathcal{F}_w(L^{p_-} - L^2)$.

Definition 2.3 ([1]). Let $p, q \in [1, \infty]$ with $p \leq q$, $w \in A_\infty(\mathbb{R}^n)$ and $\{T_t\}_{t>0}$ be a family of sublinear operators. The family $\{T_t\}_{t>0}$ is said to satisfy *weighted L^p - L^q off-diagonal estimates on balls*, denoted by $T_t \in \mathcal{O}_w(L^p - L^q)$, if there exist constants $\theta_1, \theta_2 \in [0, \infty)$ and $C, c \in (0, \infty)$ such that, for all $t \in (0, \infty)$ and all balls $B := B(x_B, r_B) \subset \mathbb{R}^n$ with $x_B \in \mathbb{R}^n$ and $r_B \in (0, \infty)$, and $f \in L^p_{\text{loc}}(w, \mathbb{R}^n)$,

$$(2.1) \quad \left\{ \frac{1}{w(B)} \int_B |T_t(\chi_B f)(x)|^q w(x) dx \right\}^{1/q} \leq C \left[\Upsilon \left(\frac{r_B}{t^{1/2}} \right) \right]^{\theta_2} \left\{ \frac{1}{w(B)} \int_B |f(x)|^p w(x) dx \right\}^{1/p}$$

and, for all $j \in \mathbb{N}$ with $j \geq 3$,

$$\left\{ \frac{1}{w(2^j B)} \int_{U_j(B)} |T_t(\chi_B f)(x)|^q w(x) dx \right\}^{1/q} \leq C 2^{j\theta_1} \left[\Upsilon \left(\frac{2^j r_B}{t^{1/2}} \right) \right]^{\theta_2} e^{-c \frac{(2^j r_B)^2}{t}} \left\{ \frac{1}{w(B)} \int_B |f(x)|^p w(x) dx \right\}^{1/p}$$

and

$$(2.2) \quad \left\{ \frac{1}{w(B)} \int_B |T_t(\chi_{U_j(B)} f)(x)|^q w(x) dx \right\}^{1/q} \leq C 2^{j\theta_1} \left[\Upsilon \left(\frac{2^j r_B}{t^{1/2}} \right) \right]^{\theta_2} e^{-c \frac{(2^j r_B)^2}{t}} \left\{ \frac{1}{w(2^j B)} \int_{U_j(B)} |f(x)|^p w(x) dx \right\}^{1/p},$$

where $U_j(B)$ is as in (1.7) and, for all $s \in (0, \infty)$,

$$\Upsilon(s) := \max \left\{ s, \frac{1}{s} \right\}.$$

By borrowing some ideas from the proof of [1, Proposition 3.2], we obtain the following conclusions.

Proposition 2.4. *Let $w \in A_\infty(\mathbb{R}^n) \cap RH_s(\mathbb{R}^n)$ with $s \in (1, \infty]$ and $\{T_t\}_{t \geq 0}$ be a family of sublinear operators.*

- (i) *If $s = (1 - \frac{p_0}{2}) \frac{p}{p-p_0}$, $1 \leq p_0 < p < 2$ and $T_t \in \mathcal{O}_w(L^{p_0} - L^2)$, then $T_t \in \mathcal{F}_w(L^p - L^2)$.*
- (ii) *If $s = \infty$ and $T_t \in \mathcal{O}_w(L^{p_0} - L^2)$ with $1 \leq p_0 < 2$, then $T_t \in \mathcal{F}_w(L^{p_0} - L^2)$.*

Proof. To show (i), let E, F be two closed sets of \mathbb{R}^n , $t \in [0, \infty)$ and $f \in L^p(w^{\frac{p}{2}}, E)$ with $\text{supp } f \subset E$. We now consider two cases.

Case 1) $d(E, F) > 0$ and $0 \leq t < [\frac{d(E, F)}{16}]^2$. In this case, let $r := \frac{d(E, F)}{16}$ and choose a family of balls, $B_k := B(x_k, r)$ with $k \in \mathbb{N}$ and $x_k \in \mathbb{R}^n$, such that, for any $k_1 \neq k_2$, $|x_{k_1} - x_{k_2}| \geq \frac{r}{2}$ and $\cup_{k \in \mathbb{N}} B_k = \mathbb{R}^n$. Observe that, if $x \in F$ and $y \in E$, then $|x - y| \geq d(E, F) = 16r$. Thus, if $x \in B_k$ for some $k \in \mathbb{N}$, then $y \notin 4B_k$, which further implies that there exists some $j \geq 3$ such that $y \in U_j(B_k)$. Let $\mathcal{A} := \{k \in \mathbb{N} : F \cap B_k \neq \emptyset\}$. By the fact that $\text{supp } f \subset E$, the Minkowski inequality, the Hölder inequality and (2.2) with $q = 2$, $p = p_0$ and $B = B_k$, we see that

$$\begin{aligned}
 (2.3) \quad \|T_t(f)\|_{L^2(w, F)}^2 &\leq \sum_{k \in \mathcal{A}} \int_{B_k} |T_t(f)(x)|^2 w(x) dx \\
 &\leq \sum_{k \in \mathcal{A}} \left\{ \sum_{j=3}^{\infty} \left[\int_{B_k} |T_t(\chi_{U_j(B_k)} f)(x)|^2 w(x) dx \right]^{\frac{1}{2}} \right\}^2 \\
 &\lesssim \sum_{k \in \mathcal{A}} \left\{ \sum_{j=3}^{\infty} 2^{j\theta_1} \left[\Upsilon \left(\frac{2^j r}{\sqrt{t}} \right) \right]^{\theta_2} e^{-c \frac{4^j r^2}{t}} [w(2^j B_k)]^{\frac{1}{2} - \frac{1}{p_0}} \right. \\
 &\quad \times \left. \left[\int_{U_j(B_k)} |f(x)|^{p_0} w(x) dx \right]^{\frac{1}{p_0}} \right\}^2 \\
 &\lesssim \sum_{k \in \mathcal{A}} \left\{ \sum_{j=3}^{\infty} 2^{j\theta_1} \left[\Upsilon \left(\frac{2^j r}{\sqrt{t}} \right) \right]^{\theta_2} e^{-c \frac{4^j r^2}{t}} [w(2^j B_k)]^{\frac{1}{2} - \frac{1}{p_0}} \right. \\
 &\quad \times \left[\int_{U_j(B_k)} |f(x)|^p [w(x)]^{\frac{p}{2}} dx \right]^{\frac{1}{p}} \\
 &\quad \times \left. \left[\int_{2^j B_k} [w(x)]^{(1 - \frac{p_0}{2}) \frac{p}{p-p_0}} dx \right]^{\frac{1}{p_0} - \frac{1}{p}} \right\}^2.
 \end{aligned}$$

Since $p \in (1, 2)$, we see that $s = (1 - \frac{p_0}{2}) \frac{p}{p-p_0} \in (1, \infty)$. From the fact $w \in RH_s(\mathbb{R}^n)$ and

the assumption $0 \leq t < r^2$, it follows that, for all $k \in \mathcal{A}$ and $j \geq 3$,

$$(2.4) \quad [w(2^j B_k)]^{\frac{1}{2} - \frac{1}{p_0}} \left[\int_{2^j B_k} [w(x)]^{(1 - \frac{p_0}{2}) \frac{p}{p-p_0}} dx \right]^{\frac{1}{p_0} - \frac{1}{p}} \lesssim |2^j B_k|^{\frac{1}{2} - \frac{1}{p}} \\ \lesssim r^{n(\frac{1}{2} - \frac{1}{p})} \lesssim t^{\frac{n}{2}(\frac{1}{2} - \frac{1}{p})}.$$

It is easy to see that there exist positive constants c_1 and c_2 such that

$$2^{j\theta_1} \left[\Upsilon \left(\frac{2^j r}{\sqrt{t}} \right) \right]^{\theta_2} e^{-c \frac{4^j r^2}{t}} \lesssim e^{-c_1 4^j} e^{-c_2 \frac{r^2}{t}}.$$

By this, (2.4), (2.3), the Hölder inequality, $2/p > 1$ and the fact that $r = \frac{d(E, F)}{16}$, we know that there exist positive constants c and \tilde{c} such that

$$(2.5) \quad \|T_t(f)\|_{L^2(w, F)}^2 \\ \lesssim \left[t^{-\frac{n}{2}(\frac{1}{p} - \frac{1}{2})} e^{-c_2 \frac{r^2}{t}} \right]^2 \sum_{k \in \mathcal{A}} \left\{ \sum_{j=3}^{\infty} e^{-c_1 4^j} \left[\int_{U_j(B_k)} |f(x)|^p [w(x)]^{\frac{p}{2}} dx \right]^{\frac{1}{p}} \right\}^2 \\ \lesssim \left[t^{-\frac{n}{2}(\frac{1}{p} - \frac{1}{2})} e^{-c_2 \frac{r^2}{t}} \right]^2 \sum_{k \in \mathcal{A}} \left\{ \sum_{j=3}^{\infty} e^{-\tilde{c} 4^j} \int_{U_j(B_k)} |f(x)|^p [w(x)]^{\frac{p}{2}} dx \right\}^{\frac{2}{p}} \\ \lesssim \left[t^{-\frac{n}{2}(\frac{1}{p} - \frac{1}{2})} e^{-c \frac{[d(E, F)]^2}{t}} \right]^2 \left\{ \sum_{k \in \mathcal{A}} \sum_{j=3}^{\infty} e^{-\tilde{c} 4^j} \int_{U_j(B_k)} |f(x)|^p [w(x)]^{\frac{p}{2}} dx \right\}^{\frac{2}{p}} \\ \lesssim \left[t^{-\frac{n}{2}(\frac{1}{p} - \frac{1}{2})} e^{-c \frac{[d(E, F)]^2}{t}} \right]^2 \left\{ \int_E \sum_{j=3}^{\infty} \sum_{k \in \mathcal{A}} e^{-\tilde{c} 4^j} \chi_{U_j(B_k)}(x) |f(x)|^p [w(x)]^{\frac{p}{2}} dx \right\}^{\frac{2}{p}}.$$

Notice that, for all $x \in \mathbb{R}^n$, there exists some $k_0 \in \mathbb{N}$ such that $x \in B_{k_0}$. Then, we know that, for all $j \geq 3$,

$$\sum_{k \in \mathcal{A}} \chi_{U_j(B_k)}(x) \leq \#\{k \in \mathbb{N} : x \in 2^j B_k\} \leq \#\{k \in \mathbb{N} : x_k \in B(x_{k_0}, 2^{j+1}r)\} \leq 2^{n(j+3)},$$

which further implies that there exists a positive constant C such that, for all $x \in \mathbb{R}^n$,

$$\sum_{j=3}^{\infty} \sum_{k \in \mathcal{A}} e^{-\tilde{c} 4^j} \chi_{U_j(B_k)}(x) \leq \sum_{j=3}^{\infty} 2^{n(j+3)} e^{-\tilde{c} 4^j} \leq C < \infty.$$

From this and (2.5), we deduce that, for all $0 \leq t < [\frac{d(E, F)}{16}]^2$,

$$(2.6) \quad \|T_t(f)\|_{L^2(w, F)} \lesssim t^{-\frac{n}{2}(\frac{1}{p} - \frac{1}{2})} e^{-c \frac{[d(E, F)]^2}{t}} \|f\|_{L^p(w^{\frac{p}{2}}, E)}.$$

Case 2) $t \geq [\frac{d(E, F)}{16}]^2$. In this case, let $r := \sqrt{t}$. We also choose a family of balls, $\{B_k\}_{k \in \mathbb{N}} = \{B(x_k, r)\}_{k \in \mathbb{N}}$, as in *Case 1*), where $k \in \mathbb{N}$ and $x_k \in \mathbb{R}^n$. Let also

$$\mathcal{A} := \{k \in \mathbb{N} : F \cap B_k \neq \emptyset\}.$$

Then, by the Minkowski inequality, we see that

$$\begin{aligned} \|T_t(f)\|_{L^2(w, F)}^2 &\leq \sum_{k \in \mathcal{A}} \int_{B_k} |T_t(f)(x)|^2 w(x) dx \\ &\leq \sum_{k \in \mathcal{A}} \left\{ \sum_{j=0}^{\infty} \left[\int_{B_k} |T_t(\chi_{U_j(B_k)} f)(x)|^2 w(x) dx \right]^{\frac{1}{2}} \right\}^2. \end{aligned}$$

For $j \in \{0, 1, 2\}$, we use (2.1) with $B = 4B_k$ to bound it. Then, by (2.2) and an argument similar to that used in the estimate of *Case 1*), we obtain

$$\|T_t(f)\|_{L^2(w, F)} \lesssim t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})} \|f\|_{L^p(w^{\frac{p}{2}}, E)} \lesssim t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})} e^{-c \frac{[d(E, F)]^2}{t}} \|f\|_{L^p(w^{\frac{p}{2}}, E)}.$$

This, together with (2.6), then completes the proof of Proposition 2.4(i).

To show (ii), we use the same method as that used in the proof of (i). In this case, we also first assume that $d(E, F) > 0$ and $0 \leq t < [\frac{d(E, F)}{16}]^2$. By the same argument as that used in (2.3), we know that

$$\begin{aligned} (2.7) \quad \|T_t(f)\|_{L^2(w, F)}^2 &\leq \sum_{k \in \mathcal{A}} \int_{B_k} |T_t(f)(x)|^2 w(x) dx \\ &\leq \sum_{k \in \mathcal{A}} \left\{ \sum_{j=3}^{\infty} \left[\int_{B_k} |T_t(\chi_{U_j(B_k)} f)(x)|^2 w(x) dx \right]^{\frac{1}{2}} \right\}^2 \\ &\lesssim \sum_{k \in \mathcal{A}} \left\{ \sum_{j=3}^{\infty} 2^{j\theta_1} \left[\Upsilon \left(\frac{2^j r}{\sqrt{t}} \right) \right]^{\theta_2} e^{-c \frac{4^j r^2}{t}} [w(2^j B_k)]^{\frac{1}{2} - \frac{1}{p_0}} \right. \\ &\quad \times \left. \left[\int_{U_j(B_k)} |f(x)|^{p_0} w(x) dx \right]^{\frac{1}{p_0}} \right\}^2 \\ &\lesssim \sum_{k \in \mathcal{A}} \left\{ \sum_{j=3}^{\infty} 2^{j\theta_1} \left[\Upsilon \left(\frac{2^j r}{\sqrt{t}} \right) \right]^{\theta_2} e^{-c \frac{4^j r^2}{t}} [w(2^j B_k)]^{\frac{1}{2} - \frac{1}{p_0}} \right. \\ &\quad \times \left. \left[\int_{U_j(B_k)} |f(x)|^{p_0} [w(x)]^{\frac{p_0}{2}} dx \right]^{\frac{1}{p_0}} \|w\|_{L^\infty(2^j B_k)}^{\frac{1}{p_0} - \frac{1}{2}} \right\}^2. \end{aligned}$$

From $w \in RH_\infty(\mathbb{R}^n)$ and $0 \leq t < r^2$, it follows that, for all $k \in \mathcal{A}$ and $j \geq 3$,

$$\|w\|_{L^\infty(2^j B_k)}^{\frac{1}{p_0} - \frac{1}{2}} [w(2^j B_k)]^{\frac{1}{2} - \frac{1}{p_0}} \lesssim |2^j B_k|^{\frac{1}{2} - \frac{1}{p_0}} \lesssim r^{n(\frac{1}{2} - \frac{1}{p_0})} \lesssim t^{\frac{n}{2}(\frac{1}{2} - \frac{1}{p_0})}.$$

By this, (2.7) and an argument similar to that used in the proof of *Case 1*) of (i), we see that, for all $0 \leq t < [\frac{d(E, F)}{16}]^2$,

$$(2.8) \quad \|T_t(f)\|_{L^2(w, F)} \lesssim t^{-\frac{n}{2}(\frac{1}{p_0}-\frac{1}{2})} e^{-c \frac{[d(E, F)]^2}{t}} \|f\|_{L^{p_0}(w^{\frac{p_0}{2}}, E)}.$$

When $t \geq [\frac{d(E,F)}{16}]^2$, by an argument similar to that used in the proof of *Case 2*) of (i), we know (2.8) also holds true. This finishes the proof of Proposition 2.4(ii) and hence Proposition 2.4. \square

Remark 2.5. From [16, Theorems 1 and 5], we deduce that, if the matrix A associated with L_w (see (1.3)) is real symmetric, then $\{e^{-tL_w}\}_{t \geq 0}$ and $\{tL_w e^{-tL_w}\}_{t \geq 0}$ have heat kernels. Moreover, the heat kernels both satisfy the weighted Gaussian bounds (see [16, p. 1 (2)]). By [1, Proposition 2.2], we know that the weighted Gaussian bounds ([16, p. 1 (2)]) is equivalent to the weighted L^1 - L^∞ off-diagonal estimates on balls. Since $\mathcal{O}_w(L^1 - L^\infty) \subset \mathcal{O}_w(L^p - L^q)$ with $1 \leq p \leq q \leq \infty$ (see [1, Comments 4]), it follows that, if the matrix A is real symmetric, then, for any $p_0 \in [1, 2)$, $tL_w e^{-tL_w} \in \mathcal{O}_w(L^{p_0} - L^2)$. From Proposition 2.4, we further deduce that, for any $p \in [1, 2)$, $tL_w e^{-tL_w} \in \mathcal{F}_w(L^p - L^2)$.

Generally, we have the following conclusion.

Proposition 2.6. *For $n \geq 3$, let $w^{-1} \in A_{2-\frac{2}{n}}(\mathbb{R}^n)$, $p_- = \frac{2n}{n+2}$ and L_w be the degenerate elliptic operator satisfying (1.1) and (1.2). Then there exist positive constants C and \tilde{C} such that, for all closed sets E and F , $t \in (0, \infty)$ and $f \in L^{p_-}(w^{\frac{p_-}{2}}, \mathbb{R}^n)$ supported in E ,*

$$(2.9) \quad \|e^{-tL_w}(f)\|_{L^2(w, F)} \leq Ct^{-\frac{n}{2}(\frac{1}{p_-}-\frac{1}{2})} e^{-\tilde{C}\frac{[d(E,F)]^2}{t}} \|f\|_{L^{p_-}(w^{\frac{p_-}{2}}, E)}.$$

The proof of Proposition 2.6 relies on an exponential perturbation method from [17] and the boundedness of the Riesz potential in weighted Lebesgue spaces from [35]. We first introduce some notions and lemmas.

Let $\mathcal{E}(\mathbb{R}^n)$ be the set of all bounded real-valued functions $\phi \in C^\infty(\mathbb{R}^n)$ such that, for all multi-indices $\alpha \in (\mathbb{Z}_+)^n$ and $|\alpha| = 1$, $\|\partial^\alpha \phi\|_{L^\infty(\mathbb{R}^n)} \leq 1$. Now, for $\nu \in \mathbb{R}_+ := (0, \infty)$ and $\phi \in \mathcal{E}(\mathbb{R}^n)$, let

$$(2.10) \quad L_{\nu, \phi} := e^{\nu\phi} L_w e^{-\nu\phi}.$$

For all $f, g \in \mathcal{H}_0^1(w, \mathbb{R}^n)$, the *twist sesquilinear form* $\mathfrak{a}_{\nu, \phi}$ is defined by setting

$$\mathfrak{a}_{\nu, \phi}(f, g) := \int_{\mathbb{R}^n} \left[A(x) \nabla(e^{-\nu\phi} f)(x) \right] \cdot \nabla(e^{\nu\phi} g)(x) dx.$$

Then, by the definition of L_w , we know that

$$(2.11) \quad \mathfrak{a}_{\nu, \phi}(f, g) = (L_{\nu, \phi}(f), g)_{L^2(w, \mathbb{R}^n)} := \int_{\mathbb{R}^n} L_{\nu, \phi}(f)(x) \overline{g(x)} w(x) dx.$$

Let $\{e^{-tL_{\nu, \phi}}\}_{t \geq 0}$ be the heat semigroup generated by $L_{\nu, \phi}$.

Notice that the conditions (1.1) and (1.2) imply that L_w is of type ω , where $\omega := \arctan(\Lambda/\lambda) \in [0, \frac{\pi}{2})$; see [34] (also [13, p. 293]) for the details. Hence, for $z \in \Sigma(\pi/2 - \omega)$, where

$$\Sigma(\pi/2 - \omega) := \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \pi/2 - \omega\},$$

it holds true that

$$e^{-zL_w}(f) = \frac{1}{2\pi i} \int_{\Gamma} e^{z\xi} (\xi I + L_w)^{-1}(f) d\xi,$$

where

$$\Gamma := \gamma^+ \cup \gamma^- := \left\{ z \in \mathbb{C} : z = r^{i\theta}, r \in (0, \infty) \right\} \cup \left\{ z \in \mathbb{C} : z = r^{-i\theta}, r \in (0, \infty) \right\}$$

for some $\theta \in (\pi/2 + |\arg(z)|, \pi - \omega)$. This, together with (2.10), implies that, for all $t \in (0, \infty)$,

$$(2.12) \quad e^{-tL_{\nu, \phi}} = e^{\nu\phi} e^{-tL_w} e^{-\nu\phi}.$$

The following two lemmas are, respectively, [43, Lemma 2.4] and [43, Lemma 2.5].

Lemma 2.7 ([43]). *Let $w \in A_2(\mathbb{R}^n)$ and L_w be the degenerate elliptic operator satisfying the degenerate elliptic conditions (1.1) and (1.2). Then there exists a positive constant C such that, for all $\nu \in \mathbb{R}_+$, $\phi \in \mathcal{E}(\mathbb{R}^n)$ and $f \in \mathcal{H}_0^1(w, \mathbb{R}^n)$,*

$$(2.13) \quad |\mathfrak{a}_{\nu, \phi}(f, f) - \mathfrak{a}(f, f)| \leq \frac{1}{4} \Re \{ \mathfrak{a}(f, f) \} + C\nu^2 \|f\|_{L^2(w, \mathbb{R}^n)}^2.$$

Lemma 2.8 ([43]). *Let $w \in A_2(\mathbb{R}^n)$, $k \in \mathbb{Z}_+$ and L_w be the degenerate elliptic operator satisfying the degenerate elliptic conditions (1.1) and (1.2). Then there exist positive constants C_0 and C_1 such that, for all $\nu \in \mathbb{R}_+$, $\phi \in \mathcal{E}(\mathbb{R}^n)$, $t \in (0, \infty)$ and $f \in L^2(w, \mathbb{R}^n)$,*

$$(2.14) \quad \left\| (tL_{\nu, \phi})^k e^{-tL_{\nu, \phi}}(f) \right\|_{L^2(w, \mathbb{R}^n)} \leq C_0 e^{C_1 \nu^2 t} \|f\|_{L^2(w, \mathbb{R}^n)}.$$

Let $1 < p < q < \infty$. Recall also the following definition of $A_{p,q}(\mathbb{R}^n)$ weights from [35]. A nonnegative and locally integrable function w is said to belong to the *weight class* $A_{p,q}(\mathbb{R}^n)$, if

$$[w]_{A_{p,q}(\mathbb{R}^n)} := \sup_{B \subset \mathbb{R}^n} \left\{ \frac{1}{|B|} \int_B [w(x)]^q dx \right\}^{\frac{1}{q}} \left\{ \frac{1}{|B|} \int_B [w(x)]^{-p'} dx \right\}^{\frac{1}{p'}} < \infty,$$

where the supremum is taken over all open balls $B \subset \mathbb{R}^n$ and $p' := \frac{p}{p-1}$ denotes the *conjugate exponent* of p .

We are now in a position to prove Proposition 2.6.

Proof of Proposition 2.6. Let $p_+ := \frac{2n}{n-2}$. It is easy to see that $\frac{1}{p_-} + \frac{1}{p_+} = 1$ and

$$(2.15) \quad \frac{1}{2} - \frac{1}{p_+} = \frac{1}{p_-} - \frac{1}{2} = \frac{1}{n}.$$

It is well known that $w \in A_{p,q}(\mathbb{R}^n)$ if and only if $w^{-p'} \in A_{1+\frac{p'}{q}}(\mathbb{R}^n)$ (see [35, pp. 266-267]), where $1 < p < q < \infty$. Hence, $w^{-1} \in A_{2-\frac{2}{n}}(\mathbb{R}^n)$ is equivalent to $w^{1/2} \in A_{2,p_+}(\mathbb{R}^n)$. Then, by [35, Theorem 4], we know that the Riesz potential

$$(-\Delta)^{-1/2}(f)(x) := \frac{1}{\gamma(1)} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-1}} dy,$$

where $x \in \mathbb{R}^n$ and $\gamma(1) = 2\pi^{\frac{n}{2}}\Gamma(\frac{1}{2})/\Gamma(\frac{n-1}{2})$, is bounded from $L^2(w, \mathbb{R}^n)$ to $L^{p+}(w^{\frac{p+}{2}}, \mathbb{R}^n)$. Therefore, for all $g \in L^2(w, \mathbb{R}^n)$,

$$\left\| (-\Delta)^{-\frac{1}{2}}(g) \right\|_{L^{p+}(w^{\frac{p+}{2}}, \mathbb{R}^n)} \lesssim \|g\|_{L^2(w, \mathbb{R}^n)}.$$

Moreover, since, for any $u \in C_c^\infty(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, it holds true that

$$(2.16) \quad |u(x)| \lesssim \int_{\mathbb{R}^n} \frac{|\nabla u(y)|}{|x-y|^{n-1}} dy \lesssim (-\Delta)^{-\frac{1}{2}}(|\nabla u|)(x)$$

(see [37, p.125]), this, combined with a density argument, implies that, for all $h \in \mathcal{H}_0^1(w, \mathbb{R}^n)$,

$$(2.17) \quad \|h\|_{L^{p+}(w^{\frac{p+}{2}}, \mathbb{R}^n)} \lesssim \|\nabla h\|_{L^2(w, \mathbb{R}^n)}.$$

Now, for all $t, \nu \in (0, \infty)$, $\phi \in \mathcal{E}(\mathbb{R}^n)$ and $f \in L^2(w, \mathbb{R}^n)$, let $f_t := e^{-tL_{\nu, \phi}}(f)$. By (2.17) and the degenerate elliptic condition (1.2), we obtain

$$(2.18) \quad \|f_t\|_{L^{p+}(w^{\frac{p+}{2}}, \mathbb{R}^n)} \lesssim \|\nabla(f_t)\|_{L^2(w, \mathbb{R}^n)} \lesssim [\Re\{\mathfrak{a}(f_t, f_t)\}]^{1/2}.$$

From Lemma 2.7, it follows that

$$\begin{aligned} \Re\{\mathfrak{a}(f_t, f_t)\} &\leq |\Re\{\mathfrak{a}(f_t, f_t)\} - \Re\{\mathfrak{a}_{\nu, \phi}(f_t, f_t)\}| + |\Re\{\mathfrak{a}_{\nu, \phi}(f_t, f_t)\}| \\ &\leq \frac{1}{4}\Re\{\mathfrak{a}(f_t, f_t)\} + C\nu^2\|f_t\|_{L^2(w, \mathbb{R}^n)}^2 + |\mathfrak{a}_{\nu, \phi}(f_t, f_t)|, \end{aligned}$$

where the positive constant C is as in Lemma 2.7. This, together with (2.18), (2.11), the Hölder inequality and Lemma 2.8, implies that there exists a positive constant M_0 such that

$$\begin{aligned} (2.19) \quad \|f_t\|_{L^{p+}(w^{\frac{p+}{2}}, \mathbb{R}^n)} &\lesssim \left[\nu^2 \|f_t\|_{L^2(w, \mathbb{R}^n)}^2 + |\mathfrak{a}_{\nu, \phi}(f_t, f_t)| \right]^{1/2} \\ &\lesssim \left[\nu^2 \|f_t\|_{L^2(w, \mathbb{R}^n)}^2 + \|L_{\nu, \phi}(f_t)\|_{L^2(w, \mathbb{R}^n)} \|f_t\|_{L^2(w, \mathbb{R}^n)} \right]^{1/2} \\ &\lesssim \left[\nu^2 e^{2C_1\nu^2 t} \|f\|_{L^2(w, \mathbb{R}^n)}^2 + \frac{1}{t} e^{2C_1\nu^2 t} \|f\|_{L^2(w, \mathbb{R}^n)}^2 \right]^{1/2} \\ &\lesssim t^{-1/2} e^{M_0\nu^2 t} \|f\|_{L^2(w, \mathbb{R}^n)}, \end{aligned}$$

where the positive constant C_1 is as in Lemma 2.8 and the implicit positive constants are independent of t, ν and f .

Take $\phi \in \mathcal{E}(\mathbb{R}^n)$ satisfying $\phi|_E \geq 0$ and $\phi|_F \leq -\frac{d(E, F)}{1+\epsilon}$, where ϵ is some suitable positive constant. By this, (2.12) and (2.19), we see that, for all $g \in L^2(w, F)$ supported in F ,

$$\begin{aligned} (2.20) \quad \|e^{-tL_w}(g)\|_{L^{p+}(w^{\frac{p+}{2}}, E)} &= \left\| e^{-\nu\phi} e^{\nu\phi} e^{-tL_w} \left(e^{-\nu\phi} e^{\nu\phi} g \right) \right\|_{L^{p+}(w^{\frac{p+}{2}}, E)} \\ &\leq \left\| e^{-tL_{\nu, \phi}} \left(e^{\nu\phi} g \right) \right\|_{L^{p+}(w^{\frac{p+}{2}}, E)} \end{aligned}$$

$$\begin{aligned}
&\lesssim t^{-1/2} e^{M_0 \nu^2 t} \left\| e^{\nu \phi} g \right\|_{L^2(w, F)} \\
&\lesssim t^{-1/2} e^{M_0 \nu^2 t} e^{-\nu \frac{d(E, F)}{1+\epsilon}} \|g\|_{L^2(w, F)},
\end{aligned}$$

where the positive constant M_0 is as in (2.19). This, combined with the choice that $\nu := \frac{d(E, F)}{\widetilde{C}_0 t}$ with $\widetilde{C}_0 > (1 + \epsilon)M_0$, implies that there exists a positive constant K_0 such that, for all $g \in L^2(w, F)$ supported in F ,

$$\begin{aligned}
(2.21) \quad \left\| e^{-tL_w}(g) \right\|_{L^{p_+}(w^{\frac{p_+}{2}}, E)} &\lesssim t^{-1/2} e^{-[\frac{1}{\widetilde{C}_0}(\frac{1}{1+\epsilon} - \frac{M_0}{\widetilde{C}_0})] \frac{[d(E, F)]^2}{t}} \|g\|_{L^2(w, F)} \\
&\sim t^{-1/2} e^{-K_0 \frac{[d(E, F)]^2}{t}} \|g\|_{L^2(w, F)}.
\end{aligned}$$

Using duality, the Hölder inequality, (2.21) and (2.15), we conclude that, for all $f \in L^{p_-}(w^{\frac{p_-}{2}}, E)$ supported in E and $g \in L^2(w, F)$ supported in F ,

$$\begin{aligned}
&\left| \int_F e^{-tL_w^*}(f)(x) \overline{g(x)} w(x) dx \right| \\
&= \left| \int_E f(x) \overline{e^{-tL_w}(g)(x)} [w(x)]^{\frac{p_+}{2p_+} + \frac{1}{2}} dx \right| \\
&\leq \left\{ \int_E |e^{-tL_w}(g)(x)|^{p_+} [w(x)]^{\frac{p_+}{2}} dx \right\}^{1/p_+} \left\{ \int_E |f(x)|^{p_-} [w(x)]^{\frac{p_-}{2}} dx \right\}^{1/p_-} \\
&\lesssim t^{-\frac{n}{2}(\frac{1}{p_-} - \frac{1}{2})} e^{-K_0 \frac{[d(E, F)]^2}{t}} \|g\|_{L^2(w, F)} \|f\|_{L^{p_-}(w^{\frac{p_-}{2}}, E)},
\end{aligned}$$

where the positive constant K_0 is as in (2.21). By this and the dual representation of the $L^2(w, F)$ norm of $e^{-tL_w^*}(f)$, we see that

$$\left\| e^{-tL_w^*}(f) \right\|_{L^2(w, F)} \lesssim t^{-\frac{n}{2}(\frac{1}{p_-} - \frac{1}{2})} e^{-K_0 \frac{[d(E, F)]^2}{t}} \|f\|_{L^{p_-}(w^{\frac{p_-}{2}}, E)}.$$

Observing the above estimates also hold true via replacing $e^{-tL_w^*}$ by e^{-tL_w} , we then complete the proof of Proposition 2.6. \square

2.3 Weighted tent spaces

For all measurable functions f on \mathbb{R}_+^{n+1} and $x \in \mathbb{R}^n$, let

$$A(f)(x) := \left[\iint_{\Gamma(x)} |f(y, t)|^2 \frac{dy dt}{t^{n+1}} \right]^{1/2},$$

where $\Gamma(x)$ is as in (1.4) with $\alpha = 1$. For all $p \in (0, \infty)$ and $w \in A_\infty(\mathbb{R}^n)$, the *weighted tent space* $T_w^p(\mathbb{R}_+^{n+1})$ is defined to be the space of all measurable functions f such that $\|f\|_{T_w^p(\mathbb{R}_+^{n+1})} := \|A(f)\|_{L^p(w, \mathbb{R}^n)} < \infty$. When $w \equiv 1$, the space $T_w^p(\mathbb{R}_+^{n+1})$ was studied in [10] and is simply denoted by $T^p(\mathbb{R}_+^{n+1})$.

For any open set $O \subset \mathbb{R}^n$, the tent over O is defined by

$$\widehat{O} := \{(x, t) \in \mathbb{R}_+^{n+1} : \text{dist}(x, O^c) \geq t\}.$$

Let $p \in (0, 1]$ and $w \in A_\infty(\mathbb{R}^n)$. A measurable function a on \mathbb{R}_+^{n+1} is called a (w, p, ∞) -atom if there exists a ball $B \subset \mathbb{R}^n$ such that

- (i) $\text{supp } a \subset \widehat{B}$;
- (ii) for all $q \in (1, \infty)$,

$$\|a\|_{T_w^p(\mathbb{R}_+^{n+1})} := \left\{ \int_{\mathbb{R}^n} \left[\iint_{\Gamma(x)} |a(y, t)|^2 \frac{dy dt}{t^{n+1}} \right]^{\frac{q}{2}} dx \right\}^{\frac{1}{q}} \leq |B|^{\frac{1}{q}} [w(B)]^{-\frac{1}{p}}.$$

Remark 2.9. (i) Every (w, p, ∞) -atom a belongs to $T_w^p(\mathbb{R}_+^{n+1})$ and $\|a\|_{T_w^p(\mathbb{R}_+^{n+1})} \leq C$, where the positive constant C is independent of a (see [4, p. 7]).

- (ii) If $\text{supp } f \subset \widehat{B}$ for some ball $B \subset \mathbb{R}^n$, then $\text{supp } A(f) \subset B$.

The following lemma is needed in the proof of Theorem 1.8.

Lemma 2.10. *Let a be a (w, p, ∞) -atom, with $w \in A_\infty(\mathbb{R}^n)$ and $p \in (0, 1]$, and $\text{supp } a \subset \widehat{B}$. Then, for any $p_1 \in (1, \infty)$, there exists a positive constant C , independent of a , such that*

$$\|a\|_{T_w^{p_1}(\mathbb{R}_+^{n+1})} \leq C[w(B)]^{\frac{1}{p_1} - \frac{1}{p}}.$$

Proof. Since $w \in A_\infty(\mathbb{R}^n)$, by Lemma 2.1(ii), we know that there exists some $r \in (1, \infty)$ such that $w \in RH_{r'}(\mathbb{R}^n)$, where $1/r + 1/r' = 1$. From this, Remark 2.9(ii) and the Hölder inequality, it follows that

$$\begin{aligned} \|a\|_{T_w^{p_1}(\mathbb{R}_+^{n+1})} &= \left\{ \int_{\mathbb{R}^n} \left[\iint_{\Gamma(x)} |a(y, t)|^2 \frac{dy dt}{t^{n+1}} \right]^{\frac{p_1}{2}} w(x) dx \right\}^{\frac{1}{p_1}} \\ &\leq \left\{ \int_B \left[\iint_{\Gamma(x)} |a(y, t)|^2 \frac{dy dt}{t^{n+1}} \right]^{\frac{rp_1}{2}} dx \right\}^{\frac{1}{rp_1}} \left\{ \int_B [w(x)]^{r'} dx \right\}^{\frac{1}{r'p_1}} \\ &\lesssim |B|^{\frac{1}{rp_1}} [w(B)]^{-\frac{1}{p}} |B|^{\frac{1}{r'p_1} - \frac{1}{p_1}} [w(B)]^{\frac{1}{p_1}} \lesssim [w(B)]^{\frac{1}{p_1} - \frac{1}{p}}, \end{aligned}$$

which completes the proof of Lemma 2.10. \square

An important result concerning weighted tent spaces is that each function in $T_w^p(\mathbb{R}_+^{n+1})$ has an atomic decomposition. More precisely, we have the following result, which is a slight variant of [4, Theorem 2.6].

Lemma 2.11 ([4]). *Let $p \in (0, 1]$, $w \in A_\infty(\mathbb{R}^n)$ and $f \in T_w^p(\mathbb{R}_+^{n+1})$. Then there exist a sequence of (w, p, ∞) -atoms, $\{a_j\}_{j \in \mathbb{N}}$, and a sequence of numbers, $\{\lambda_j\}_{j \in \mathbb{N}} \subset \mathbb{C}$, such that*

$$(2.22) \quad f = \sum_{j \in \mathbb{N}} \lambda_j a_j,$$

where the series converges in $T_w^p(\mathbb{R}_+^{n+1})$. Moreover, there exist positive constants \tilde{C} and C , independent of f , such that

$$\tilde{C}\|f\|_{T_w^p(\mathbb{R}_+^{n+1})} \leq \left\{ \sum_{j \in \mathbb{N}} |\lambda_j|^p \right\}^{1/p} \leq C\|f\|_{T_w^p(\mathbb{R}_+^{n+1})}.$$

Furthermore, if $f \in T_w^p(\mathbb{R}_+^{n+1}) \cap T_w^2(\mathbb{R}_+^{n+1})$, then the series in (2.22) converges in both $T_w^p(\mathbb{R}_+^{n+1})$ and $T_w^2(\mathbb{R}_+^{n+1})$.

Proof. By [4, Theorem 2.6], we only need to show that $\|f\|_{T_w^p(\mathbb{R}_+^{n+1})} \lesssim \{\sum_{j \in \mathbb{N}} |\lambda_j|^p\}^{1/p}$ and the last conclusion of this lemma, concerning the $T_w^2(\mathbb{R}_+^{n+1})$ convergence of the series in (2.22). For all $N \in \mathbb{N}$, let

$$S_N := \sum_{j=1}^N \lambda_j a_j.$$

From Remark 2.9(i), it follows that $\{S_N\}_{N \in \mathbb{N}}$ is a Cauchy sequence in $T_w^p(\mathbb{R}_+^{n+1})$ and $\|S_N\|_{T_w^p(\mathbb{R}_+^{n+1})} \lesssim \{\sum_{j \in \mathbb{N}} |\lambda_j|^p\}^{1/p}$. Since S_N converges to f in $T_w^p(\mathbb{R}_+^{n+1})$ as $N \rightarrow \infty$, we find that $\|f\|_{T_w^p(\mathbb{R}_+^{n+1})} \lesssim \{\sum_{j \in \mathbb{N}} |\lambda_j|^p\}^{1/p}$.

Similar to the proof of [29, Proposition 3.1] (see also the proof of [27, Proposition 3.32]), we further conclude that, if $f \in T_w^p(\mathbb{R}_+^{n+1}) \cap T_w^2(\mathbb{R}_+^{n+1})$, then the series in (2.22) converges in both $T_w^p(\mathbb{R}_+^{n+1})$ and $T_w^2(\mathbb{R}_+^{n+1})$, which completes the proof of Lemma 2.11. \square

3 Proof of Theorem 1.8

In this section, we give the proof of Theorem 1.8. To this end, we first introduce some technical lemmas.

The following lemma is a well known result (see, for example, [40, Theorem 2, p. 87]).

Lemma 3.1. *Let $\Phi \in \mathcal{S}(\mathbb{R}^n)$ satisfy $\int_{\mathbb{R}^n} \Phi(x) dx = 0$, $\Phi_t(x) := \frac{1}{t^n} \Phi(\frac{x}{t})$ for all $x \in \mathbb{R}^n$ and $t \in (0, \infty)$, and $w \in A_2(\mathbb{R}^n)$. The Littlewood-Paley g -function g_Φ and square function S_Φ are defined, respectively, by setting, for all $f \in \mathcal{S}'(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,*

$$g_\Phi(f)(x) := \left[\int_0^\infty |f * \Phi_t(x)|^2 \frac{dt}{t} \right]^{1/2}$$

and

$$S_\Phi(f)(x) := \left[\iint_{\Gamma(x)} |f * \Phi_t(y)|^2 \frac{dy dt}{t^{n+1}} \right]^{1/2}.$$

Then g_Φ and S_Φ are bounded on $L^2(w, \mathbb{R}^n)$.

Remark 3.2. Let $w \in A_2(\mathbb{R}^n)$. Then it is easy to show that, via the pairing $\langle f, g \rangle := \int_{\mathbb{R}^n} f(x)g(x) dx$, where $f \in L^2(w, \mathbb{R}^n)$ and $g \in L^2(w^{-1}, \mathbb{R}^n)$, $L^2(w^{-1}, \mathbb{R}^n)$ and the dual space of $L^2(w, \mathbb{R}^n)$ coincide with equivalent norms.

The following lemma plays a key role in the proof of Theorem 1.8.

Lemma 3.3. *Let $\Phi \in \mathcal{S}(\mathbb{R}^n)$ satisfy $\int_{\mathbb{R}^n} \Phi(x) dx = 0$, $\Phi_t(x) = \frac{1}{t^n} \Phi(\frac{x}{t})$ for all $x \in \mathbb{R}^n$ and $t \in (0, \infty)$, and $w \in A_2(\mathbb{R}^n)$. For any $a \in T_w^2(\mathbb{R}_+^{n+1})$ and $x \in \mathbb{R}^n$, let*

$$(3.1) \quad \pi_\Phi(a)(x) := \int_0^\infty (a(\cdot, t) * \Phi_t)(x) \frac{dt}{t}.$$

Then π_Φ is bounded from $T_w^2(\mathbb{R}_+^{n+1})$ to $L^2(w, \mathbb{R}^n)$.

Proof. Fix any $a \in T_w^2(\mathbb{R}_+^{n+1})$ and let $\tilde{\Phi}(x) := \Phi(-x)$ for all $x \in \mathbb{R}^n$. Then, for any $f \in L^2(w^{-1}, \mathbb{R}^n)$ with $\|f\|_{L^2(w^{-1}, \mathbb{R}^n)} = 1$, by the Fubini theorem, we see that

$$(3.2) \quad \begin{aligned} \langle \pi_\Phi(a), f \rangle &= \int_{\mathbb{R}^n} \pi_\Phi(a)(x) f(x) dx = \int_0^\infty \int_{\mathbb{R}^n} (a(\cdot, t) * \Phi_t)(x) f(x) dx \frac{dt}{t} \\ &= \int_0^\infty \int_{\mathbb{R}^n} a(y, t) (\tilde{\Phi}_t * f)(y) dy \frac{dt}{t} \\ &\sim \int_0^\infty \int_{\mathbb{R}^n} \int_{B(y, t)} a(y, t) (\tilde{\Phi}_t * f)(y) \frac{dx}{t^n} dy \frac{dt}{t} \\ &\sim \int_{\mathbb{R}^n} \iint_{\Gamma(x)} a(y, t) (\tilde{\Phi}_t * f)(y) \frac{dy dt}{t^{n+1}} dx. \end{aligned}$$

Since $w \in A_2(\mathbb{R}^n)$ is equivalent to $w^{-1} \in A_2(\mathbb{R}^n)$, by (3.2), the Hölder inequality and Lemma 3.1, we find that, for all $f \in L^2(w^{-1}, \mathbb{R}^n)$ with $\|f\|_{L^2(w^{-1}, \mathbb{R}^n)} = 1$,

$$\begin{aligned} |\langle \pi_\Phi, g \rangle| &\lesssim \int_{\mathbb{R}^n} \left[\iint_{\Gamma(x)} |a(y, t)|^2 \frac{dy dt}{t^{n+1}} \right]^{\frac{1}{2}} \left[\iint_{\Gamma(x)} |(\tilde{\Phi}_t * f)(y)|^2 \frac{dy dt}{t^{n+1}} \right]^{\frac{1}{2}} dx \\ &\lesssim \|A(a)\|_{L^2(w, \mathbb{R}^n)} \|g_{\tilde{\Phi}}(f)\|_{L^2(w^{-1}, \mathbb{R}^n)} \lesssim \|a\|_{T_w^2(\mathbb{R}_+^{n+1})}. \end{aligned}$$

From Remark 3.2, we further deduce that $\|\pi(a)\|_{L^2(w, \mathbb{R}^n)} \lesssim \|a\|_{T_w^2(\mathbb{R}_+^{n+1})}$, which completes the proof of Lemma 3.3. \square

By an argument similar to that used in the proof of [32, Lemma 6], we see that the following lemma holds true, the details being omitted.

Lemma 3.4. *Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Then the condition $\int_{\mathbb{R}^n} \varphi(x) dx = 0$ is equivalent to that there exist elements $\psi_k \in \mathcal{S}(\mathbb{R}^n)$, $k \in \{1, \dots, n\}$, such that*

$$\varphi = \sum_{k=1}^n \partial_k \psi_k.$$

To prove Theorem 1.8, we also need the following local weighted Sobolev imbedding theorem (see [23, Theorem (1.2)]).

Lemma 3.5 ([23]). *For any given $p \in (1, \infty)$ and $w \in A_p(\mathbb{R}^n)$, there exist positive constants C and δ such that, for all balls $B \equiv B(x_B, r_B)$ of \mathbb{R}^n with $x_B \in \mathbb{R}^n$ and $r_B \in (0, \infty)$, $u \in C_c^\infty(B)$, and numbers $k_0 \in (0, \infty)$ satisfying $1 \leq k_0 \leq \frac{n}{n-1} + \delta$,*

$$\left[\frac{1}{w(B)} \int_B |u(x)|^{k_0 p} w(x) dx \right]^{\frac{1}{k_0 p}} \leq C r_B \left[\frac{1}{w(B)} \int_B |\nabla u(x)|^p w(x) dx \right]^{\frac{1}{p}}.$$

We are now in a position to prove Theorem 1.8.

Proof of Theorem 1.8. Take $\varphi \in C_c^\infty(B(0,1))$ satisfying $\int_0^\infty t|\xi|^2|\hat{\varphi}(t\xi)|^2 dt = 1$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$ (for the existence of such functions, see [25, Lemma 1.1]). In what follows, for a function $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$, $t \in (0, \infty)$ and $x \in \mathbb{R}^n$, let $\varphi_t(x) := \frac{1}{t^n} \varphi(\frac{x}{t})$.

Let $f \in \mathcal{H}_0^1(w, \mathbb{R}^n) \cap H_w^{1,p}(\mathbb{R}^n)$, $\nabla f = (\partial_1 f, \dots, \partial_n f) =: (g_1, \dots, g_n) =: \mathbf{g}$ and, for all $(x, t) \in \mathbb{R}_+^{n+1}$, define

$$F(x, t) := t \operatorname{div}(\mathbf{g} * \varphi_t(x)) = \sum_{j=1}^n g_j * (\partial_j \varphi)_t(x).$$

By [40, Theorem 2, p. 87], we know that, for all $j \in \{1, \dots, n\}$,

$$\|S_{\partial_j \varphi}(g_j)\|_{L^p(w, \mathbb{R}^n)} \lesssim \|g_j\|_{H_w^p(\mathbb{R}^n)},$$

where $p \in (0, 1]$. This further implies that, for every $j \in \{1, \dots, n\}$, $g_j * (\partial_j \varphi)_t \in T_w^p(\mathbb{R}_+^{n+1})$. Thus, $F \in T_w^p(\mathbb{R}_+^{n+1})$ and $\|F\|_{T_w^p(\mathbb{R}_+^{n+1})} \lesssim \|\nabla f\|_{H_w^p(\mathbb{R}^n)}$.

On the other hand, noticing that, for every $j \in \{1, \dots, n\}$, the square function $S_{\partial_j \varphi}$ is bounded on $L^2(w, \mathbb{R}^n)$ (Lemma 3.1) and $g_j \in L^2(w, \mathbb{R}^n)$, we have $S_{\partial_j \varphi}(g_j) \in L^2(w, \mathbb{R}^n)$, which further implies $F \in T_w^2(\mathbb{R}_+^{n+1})$.

Thus, $F \in T_w^p(\mathbb{R}_+^{n+1}) \cap T_w^2(\mathbb{R}_+^{n+1})$. From Lemma 2.11, it follows that there exist a sequence of numbers, $\{\lambda_k\}_{k \in \mathbb{N}} \subset \mathbb{C}$, and a sequence of (w, p, ∞) -atoms, $\{\alpha_k\}_{k \in \mathbb{N}}$, such that

$$F = \sum_{k=1}^{\infty} \lambda_k \alpha_k \quad \text{in } T_w^p(\mathbb{R}_+^{n+1}) \cap T_w^2(\mathbb{R}_+^{n+1})$$

and

$$\left\{ \sum_{k=1}^{\infty} |\lambda_k|^p \right\}^{\frac{1}{p}} \sim \|F\|_{T_w^p(\mathbb{R}_+^{n+1})} \lesssim \|\nabla f\|_{H_w^p(\mathbb{R}^n)}.$$

From Lemmas 3.3 and 2.10, we deduce that, for every $j \in \{1, \dots, n\}$,

$$(3.3) \quad \pi_{\partial_j \varphi}(F) = \sum_{k=1}^{\infty} \lambda_k \pi_{\partial_j \varphi}(\alpha_k) \quad \text{in } L^2(w, \mathbb{R}^n)$$

and

$$(3.4) \quad \|\pi_{\partial_j \varphi}(\alpha_k)\|_{L^2(w, \mathbb{R}^n)} \leq C \|\alpha_k\|_{T_w^2(\mathbb{R}_+^{n+1})} \leq C [w(B_k)]^{\frac{1}{2} - \frac{1}{p}}.$$

where $\pi_{\partial_j \varphi}$ is as in (3.1) with Φ replaced by $\partial_j \varphi$ and the positive constant C is independent of k .

Since, for every $k \in \mathbb{N}$, α_k is a (w, p, ∞) -atom, we know that there exists some ball $B_k := B(x_k, r_k)$ with $x_k \in \mathbb{R}^n$ and $r_k \in (0, \infty)$ such that $\operatorname{supp} \alpha_k \subset \widehat{B_k}$ and, for every $q \in (1, \infty)$,

$$(3.5) \quad \left\{ \int_{\mathbb{R}^n} \left[\iint_{\Gamma(x)} |\alpha_k(y, t)|^2 \frac{dy dt}{t^{n+1}} \right]^{\frac{q}{2}} dx \right\}^{\frac{1}{q}} \leq |B_k|^{\frac{1}{q}} [w(B_k)]^{-\frac{1}{p}}.$$

For every $k \in \mathbb{N}$ and $x \in \mathbb{R}^n$, let $\beta_k(x) := -\int_0^\infty (\alpha_k(\cdot, t) * \varphi_t)(x) dt$ and $\tilde{B}_k := \tilde{c}B_k$, where the positive constant $\tilde{c} \in (1, \infty)$, independent of k , will be determined later. Next, we prove that, for every $k \in \mathbb{N}$,

$$\beta_k \in \mathcal{H}_0^1(w, \tilde{B}_k)$$

and

$$(3.6) \quad \mathbf{b}_k := (\pi_{\partial_1 \varphi}(\alpha_k), \dots, \pi_{\partial_n \varphi}(\alpha_k)) = \nabla \beta_k.$$

Since $\text{supp } \alpha_k \subset \widehat{B}_k$, it is easy to see $\text{supp } \beta_k \subset B_k$. By the fact that α_k is a (w, p, ∞) -atom, the Minkowski integral inequality, the Young inequality and the Hölder inequality, we further know that

$$(3.7) \quad \begin{aligned} \|\beta_k\|_{L^2(\mathbb{R}^n)} &= \left[\int_{\mathbb{R}^n} \left| \int_0^\infty (\alpha_k(\cdot, t) * \varphi_t)(x) dt \right|^2 dx \right]^{\frac{1}{2}} \\ &\leq \int_0^{r_k} \left[\int_{\mathbb{R}^n} |(\alpha_k(\cdot, t) * \varphi_t)(x)|^2 dx \right]^{\frac{1}{2}} dt \\ &\leq \int_0^{r_k} \left[\int_{\mathbb{R}^n} |\alpha_k(x, t)|^2 dx \right]^{\frac{1}{2}} \left[\int_{\mathbb{R}^n} |\varphi_t(x)| dx \right] dt \\ &\lesssim r_k \left[\int_0^{r_k} \int_{\mathbb{R}^n} |\alpha_k(x, t)|^2 \frac{dx dt}{t} \right]^{\frac{1}{2}} < \infty, \end{aligned}$$

where the last inequality follows from (3.5) with $q = 2$. Thus, $\beta_k \in L^2(\mathbb{R}^n)$.

For every $k \in \mathbb{N}$, $\delta \in (0, r_k)$ and $x \in \mathbb{R}^n$, let $F_{k,\delta}(x) := \int_\delta^\infty (\alpha_k(\cdot, t) * \varphi_t)(x) dt$. Then $\text{supp } F_{k,\delta} \subset B_k$. From an argument similar to that used in the estimate (3.7), it follows that $F_{k,\delta} \in L^2(\mathbb{R}^n)$ and

$$(3.8) \quad \lim_{\delta \rightarrow 0} \|F_{k,\delta} - \beta_k\|_{L^2(\mathbb{R}^n)} = 0.$$

Next, we prove that, for any $k \in \mathbb{N}$, $\delta \in (0, r_k)$ and almost every $x \in \mathbb{R}^n$, the partial derivatives of $F_{k,\delta}$ exist.

For any $i \in \{1, \dots, n\}$, let $\mathbf{e}_i := (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{R}^n$ be the i^{th} standard coordinate vector and $h \in (0, \infty)$. Then, we see that, for every $x \in \mathbb{R}^n$, there exists some $\theta \in (0, 1)$ such that

$$(3.9) \quad \begin{aligned} &\frac{F_{k,\delta}(x + h\mathbf{e}_i) - F_{k,\delta}(x)}{h} \\ &= \frac{1}{h} \int_\delta^\infty t \int_{\mathbb{R}^n} \alpha_k(y, t) \frac{1}{t^n} \left[\varphi\left(\frac{x + h\mathbf{e}_i - y}{t}\right) - \varphi\left(\frac{x - y}{t}\right) \right] dy \frac{dt}{t} \\ &= \int_\delta^\infty \int_{\mathbb{R}^n} \alpha_k(y, t) \frac{1}{t^n} (\partial_i \varphi) \left(\frac{x + \theta h\mathbf{e}_i - y}{t} \right) dy \frac{dt}{t}. \end{aligned}$$

Since $\varphi \in C_c^\infty(B(0, 1))$, it follows that, when $0 < h < \delta$, there exists a positive constant $C_{(\varphi)}$, depending on φ , such that

$$(3.10) \quad \left| \int_{\mathbb{R}^n} \alpha_k(y, t) \frac{1}{t^n} (\partial_i \varphi) \left(\frac{x + \theta h\mathbf{e}_i - y}{t} \right) dy \right|$$

$$\begin{aligned}
&\leq C_{(\varphi)} \int_{\mathbb{R}^n} |\alpha_k(y, t)| \frac{1}{t^n} \chi_{B(0,2)} \left(\frac{x-y}{t} \right) dy \\
&= C_{(\varphi)} \left(|\alpha_k(\cdot, t)| * (\chi_{B(0,2)})_t \right) (x) =: G(x, t).
\end{aligned}$$

By the Minkowski integral inequality, the Young inequality, the Hölder inequality, (3.5) with $q = 2$ and the fact that α_k is a (w, p, ∞) -atom, we conclude that

$$\begin{aligned}
\left\{ \int_{\mathbb{R}^n} \left[\int_{\delta}^{\infty} |G(x, t)| \frac{dt}{t} \right]^2 dx \right\}^{\frac{1}{2}} &\leq \int_{\delta}^{\infty} \left[\int_{\mathbb{R}^n} |G(x, t)|^2 dx \right]^{\frac{1}{2}} \frac{dt}{t} \\
&\leq C_{(\varphi)} \int_{\delta}^{r_k} \|\alpha_k(\cdot, t)\|_{L^2(\mathbb{R}^n)} \left\| (\chi_{B(0,2)})_t \right\|_{L^1(\mathbb{R}^n)} \frac{dt}{t} \\
&\leq C_{(\varphi, r_k, \delta)} \left[\int_0^{r_k} \int_{\mathbb{R}^n} |\alpha_k(x, t)|^2 \frac{dx dt}{t} \right]^{\frac{1}{2}} < \infty,
\end{aligned}$$

which implies that, for almost every $x \in \mathbb{R}^n$, $\int_{\delta}^{\infty} |G(x, t)| \frac{dt}{t} < \infty$. By this, (3.9), (3.10) and the dominated convergence theorem, we find that, for almost every $x \in \mathbb{R}^n$,

$$\partial_i F_{k, \delta}(x) = \int_{\delta}^{\infty} (\alpha_k(\cdot, t) * (\partial_i \varphi)_t)(x) \frac{dt}{t}.$$

Moreover, by a simple calculation, we further see that $\partial_i F_{k, \delta}$, $i \in \{1, \dots, n\}$, is just the weak derivative of $F_{k, \delta}$. From an argument similar to that used in the proof of Lemma 3.3, we conclude that, for every $k \in \mathbb{N}$, $\delta \in (0, r_k)$ and $i \in \{1, \dots, n\}$, $\partial_i F_{k, \delta} \in L^2(w, \mathbb{R}^n)$ and

$$(3.11) \quad \lim_{\delta \rightarrow 0} \|\nabla F_{k, \delta} - \mathbf{b}_k\|_{L^2(w, \mathbb{R}^n)} = 0.$$

Take $\phi \in C_c^\infty(B(0, 1))$ satisfying $\int_{\mathbb{R}^n} \phi(x) dx = 1$ and let $\phi_\varepsilon(x) := \frac{1}{\varepsilon^n} \phi(\frac{x}{\varepsilon})$ for all $x \in \mathbb{R}^n$ and $\varepsilon \in (0, \infty)$. Since, for all $k, n \in \mathbb{N}$, $\text{supp } F_{k, 1/n} \subset B_k$, $F_{k, 1/n} \in L^2(\mathbb{R}^n)$ and $\nabla F_{k, 1/n} \in L^2(w, \mathbb{R}^n)$, from [37, p. 123], [18, Theorem 2.1] and [41, Theorem 2.1.4], it follows that there exist a sequence $\{\varepsilon_n\}_{n \in \mathbb{N}}$ of positive numbers satisfying $\lim_{n \rightarrow \infty} \varepsilon_n = 0$ and a positive constant $\tilde{c} \in (1, \infty)$ such that

$$\tilde{F}_{k, 1/n} := F_{k, 1/n} * \phi_{\varepsilon_n} \in C_c^\infty(\tilde{B}_k) \quad \text{with } \tilde{B}_k = \tilde{c}B_k, \quad \nabla \tilde{F}_{k, 1/n} = (\nabla F_{k, 1/n}) * \phi_{\varepsilon_n},$$

$$(3.12) \quad \left\| \tilde{F}_{k, 1/n} - F_{k, 1/n} \right\|_{L^2(\mathbb{R}^n)} < 2^{-n}$$

and, for $w \in A_2(\mathbb{R}^n)$,

$$(3.13) \quad \left\| \nabla \tilde{F}_{k, 1/n} - \nabla F_{k, 1/n} \right\|_{L^2(w, \mathbb{R}^n)} < 2^{-n}.$$

From (3.13) and (3.11), we deduce that

$$(3.14) \quad \lim_{n \rightarrow \infty} \left\| \nabla \tilde{F}_{k, \varepsilon_n} - \mathbf{b}_k \right\|_{L^2(w, \mathbb{R}^n)} = 0.$$

By this, the fact that $\tilde{F}_{k,1/n} \in C_c^\infty(\tilde{B}_k)$ and Lemma 3.5, we know that $\{\tilde{F}_{k,1/n}\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $L^2(w, \tilde{B}_k)$ and

$$(3.15) \quad \left\| \tilde{F}_{k,1/n} \right\|_{L^2(w, \tilde{B}_k)} \leq Cr_k \left\| \nabla \tilde{F}_{k,1/n} \right\|_{L^2(w, \tilde{B}_k)},$$

where the positive constant C is independent of k or n . Therefore, there exists a function $g_k \in L^2(w, \mathbb{R}^n)$ such that

$$(3.16) \quad \lim_{n \rightarrow \infty} \left\| \tilde{F}_{k,\varepsilon_n} - g_k \right\|_{L^2(w, \mathbb{R}^n)} = 0,$$

which further implies that there exists a subsequence of $\{\tilde{F}_{k,\varepsilon_n}\}_{n \in \mathbb{N}}$ (without loss of generality, we use the same notation as the original sequence) such that, for almost every $x \in \mathbb{R}^n$, $\lim_{n \rightarrow \infty} \tilde{F}_{k,\varepsilon_n}(x) = g_k(x)$.

On the other hand, by (3.12) and (3.8), we see that $\lim_{n \rightarrow \infty} \|\tilde{F}_{k,\varepsilon_n} - \beta_k\|_{L^2(\mathbb{R}^n)} = 0$, which further implies that there exists a subsequence of $\{\tilde{F}_{k,\varepsilon_n}\}_{n \in \mathbb{N}}$ (without loss of generality, we use the same notation as the original sequence again) such that, for almost every $x \in \mathbb{R}^n$, $\lim_{n \rightarrow \infty} \tilde{F}_{k,\varepsilon_n}(x) = \beta_k(x)$.

Therefore, for every $k \in \mathbb{N}$, $\beta_k = g_k \in L^2(w, \mathbb{R}^n)$. From this, (3.14), (3.15) and (3.16), we deduce that

$$\lim_{n \rightarrow \infty} \|\tilde{F}_{k,\varepsilon_n} - \beta_k\|_{\mathcal{H}_0^1(w, \tilde{B}_k)} = 0, \quad \beta_k \in \mathcal{H}_0^1(w, \tilde{B}_k), \quad \nabla \beta_k = \mathbf{b}_k$$

and

$$\|\beta_k\|_{L^2(w, \tilde{B}_k)} \lesssim r_k \|\nabla \beta_k\|_{L^2(w, \tilde{B}_k)}.$$

This, together with (3.4), further implies that, for every $k \in \mathbb{N}$, β_k is an $H_w^{1,p}(\mathbb{R}^n)$ -atom associated to the ball \tilde{B}_k up to a harmless positive constant independent of k .

By (3.3) and (3.6), we see that

$$(3.17) \quad \sum_{k=1}^{\infty} \lambda_k \nabla \beta_k = \sum_{k=1}^{\infty} \lambda_k \mathbf{b}_k = - \int_0^{\infty} \nabla (F * \varphi_t) dt \quad \text{in } L^2(w, \mathbb{R}^n).$$

Next, we prove

$$(3.18) \quad - \int_0^{\infty} \nabla (F * \varphi_t) dt = \mathbf{g} = \nabla f \quad \text{in } L^2(w, \mathbb{R}^n).$$

Since, for any $u \in C_c^\infty(\mathbb{R}^n)$ and all $\xi \in \mathbb{R}^n$, it holds true that

$$\begin{aligned} & \left\{ - \int_0^{\infty} [t \operatorname{div} ((\nabla u) * \varphi_t)] \varphi_t dt \right\}^\wedge (\xi) \\ &= - \int_0^{\infty} \{ [t \operatorname{div} ((\nabla u) * \varphi_t)] * \varphi_t \}^\wedge (\xi) dt \\ &= - \int_0^{\infty} \left\{ t \sum_{j=1}^n \partial_j ((\partial_j u) * \varphi_t) \right\}^\wedge (\xi) \widehat{\varphi}(t\xi) dt \end{aligned}$$

$$\begin{aligned}
&= -i \int_0^\infty t \sum_{j=1}^n \xi_j ((\partial_j u) * \varphi_t)^\wedge(\xi) \widehat{\varphi}(t\xi) dt = \int_0^\infty t \sum_{j=1}^n [\xi_j \widehat{\varphi}(t\xi)]^2 \widehat{u}(\xi) dt \\
&= \widehat{u}(\xi) \int_0^\infty t [|\xi| \widehat{\varphi}(t\xi)]^2 dt = \widehat{u}(\xi),
\end{aligned}$$

then, it follows that $-\int_0^\infty [t \operatorname{div}((\nabla u) * \varphi_t)] \varphi_t dt = u$ and hence

$$-\int_0^\infty \nabla [t \operatorname{div}((\nabla u) * \varphi_t)] \varphi_t dt = \nabla u,$$

which, together with $f \in \mathcal{H}_0^1(w, \mathbb{R}^n)$ and a density argument, implies that (3.18) holds true.

Hence, from (3.17) and (3.18), it follows that

$$(3.19) \quad \nabla f = \sum_{k=1}^\infty \lambda_k \nabla \beta_k \quad \text{in } L^2(w, \mathbb{R}^n),$$

where $\{\beta_k\}_{k \in \mathbb{N}}$ is a sequence of $H_w^{1,p}(\mathbb{R}^n)$ -atoms up to a harmless positive constant.

To complete the proof of Theorem 1.8, we still need to show (1.5). From (3.19), it is easy to see $\nabla f = \sum_{k=1}^\infty \lambda_k \nabla \beta_k$ in $\mathcal{S}'(\mathbb{R}^n)$, which further implies that, for any $\eta \in \mathcal{S}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} f(x) \nabla \eta(x) dx = \sum_{k=1}^\infty \lambda_k \int_{\mathbb{R}^n} \beta_k(x) \nabla \eta(x) dx.$$

Then, by Lemma 3.4, we obtain (1.5), which completes the proof of Theorem 1.8. \square

4 Proof of Theorem 1.4

Proof of Theorem 1.4. By [43, Theorem 1.6], we know that, for $p \in (\frac{n}{n+1}, 1]$, $w \in A_{q_0}(\mathbb{R}^n)$ with $q_0 \in [1, \frac{p(n+1)}{n})$, and $f \in H_{L_w}^p(\mathbb{R}^n) \cap L^2(w, \mathbb{R}^n)$,

$$(4.1) \quad \|f\|_{H_{L_w, \text{Riesz}}^p(\mathbb{R}^n)} = \|\nabla L_w^{-1/2} f\|_{H_w^p(\mathbb{R}^n)} \lesssim \|f\|_{H_{L_w}^p(\mathbb{R}^n)},$$

which implies that

$$(4.2) \quad (H_{L_w}^p(\mathbb{R}^n) \cap L^2(w, \mathbb{R}^n)) \subset \left(H_{L_w, \text{Riesz}}^p(\mathbb{R}^n) \cap L^2(w, \mathbb{R}^n) \right).$$

Next, we prove the reverse inclusion. To this end, we only need to show that, for any $h \in H_{L_w, \text{Riesz}}^p(\mathbb{R}^n) \cap L^2(w, \mathbb{R}^n)$,

$$(4.3) \quad \|h\|_{H_{L_w}^p(\mathbb{R}^n)} \lesssim \|\nabla L_w^{-1/2} h\|_{H_w^p(\mathbb{R}^n)}.$$

Let $f := L_w^{-1/2} h$. Then, by [31, p. 281, Theorem 3.35] and [15, Theorem 1.1], we see that $f \in \mathcal{H}_0^1(w, \mathbb{R}^n)$ and $\|\nabla f\|_{L^2(w, \mathbb{R}^n)} \sim \|L_w^{1/2} f\|_{L^2(w, \mathbb{R}^n)}$. For any $x \in \mathbb{R}^n$, let

$$S_1(h)(x) := \left[\iint_{\Gamma(x)} \left| t \sqrt{L_w} e^{-t^2 L_w} h(y) \right|^2 w(y) \frac{dy}{w(B(x, t))} \frac{dt}{t} \right]^{1/2}.$$

Similar to proofs of [27, Proposition 4.9 and Corollary 4.17], we conclude that

$$\|S_1(h)\|_{L^p(w, \mathbb{R}^n)} \sim \|h\|_{H_{L_w}^p(\mathbb{R}^n)}.$$

Therefore, to prove (4.3), it suffices to show

$$(4.4) \quad \left\| S_1 \left(\sqrt{L_w}(f) \right) \right\|_{L^p(w, \mathbb{R}^n)} \lesssim \|\nabla f\|_{H_w^p(\mathbb{R}^n)}.$$

Since $f \in \mathcal{H}_0^1(w, \mathbb{R}^n)$ and $\nabla f \in H_w^p(\mathbb{R}^n)$, we see that $f \in \mathcal{H}_0^1(w, \mathbb{R}^n) \cap H_w^{1,p}(\mathbb{R}^n)$. Then, by Theorem 1.8, there exist a sequence of numbers, $\{\lambda_k\}_{k \in \mathbb{N}} \subset \mathbb{C}$, and a sequence of $H_w^{1,p}(\mathbb{R}^n)$ -atoms, $\{\beta_k\}_{k \in \mathbb{N}}$, such that

$$(4.5) \quad \nabla f = \sum_{k=1}^{\infty} \lambda_k \nabla \beta_k \quad \text{in } L^2(w, \mathbb{R}^n)$$

and

$$\left\{ \sum_{k=1}^{\infty} |\lambda_k|^p \right\}^{\frac{1}{p}} \lesssim \|\nabla f\|_{H_w^p(\mathbb{R}^n)}.$$

We claim that, to prove (4.4), it suffices to show that, for any $H_w^{1,p}(\mathbb{R}^n)$ -atom a , there exists a positive constant C , independent of a , such that

$$(4.6) \quad \left\| S_1 \left(\sqrt{L_w}(a) \right) \right\|_{L^p(w, \mathbb{R}^n)} \leq C.$$

Indeed, since L_w has a bounded H_∞ calculus in $L^2(w, \mathbb{R}^n)$ (see [13] and [34]), by [5, p. 487] (see also [34, 30]), we know that S_1 is bounded on $L^2(w, \mathbb{R}^n)$. Therefore, by this and [15, Theorem 1.1], we have

$$(4.7) \quad \left\| S_1 \left(\sqrt{L_w}(f) \right) \right\|_{L^2(w, \mathbb{R}^n)} \lesssim \left\| \sqrt{L_w}(f) \right\|_{L^2(w, \mathbb{R}^n)} \sim \|\nabla f\|_{L^2(w, \mathbb{R}^n)}.$$

From (4.7) and (4.5), we deduce that

$$\lim_{N \rightarrow \infty} \left\| S_1 \left(\sqrt{L_w}(f) \right) - S_1 \left(\sqrt{L_w} \left(\sum_{k=1}^N \lambda_k \beta_k \right) \right) \right\|_{L^2(w, \mathbb{R}^n)} = 0.$$

Hence, there exists a subsequence of $\{S_1(\sqrt{L_w}(\sum_{k=1}^N \lambda_k \beta_k))\}_{N=1}^\infty$ (without loss of generality, we use the same notation as the original sequence) such that, for almost every $x \in \mathbb{R}^n$,

$$\lim_{N \rightarrow \infty} S_1 \left(\sqrt{L_w} \left(\sum_{k=1}^N \lambda_k \beta_k \right) \right) (x) = S_1 \left(\sqrt{L_w}(f) \right) (x).$$

From this and the Minkowski inequality, we deduce that, for almost every $x \in \mathbb{R}^n$,

$$S_1 \left(\sqrt{L_w}(f) \right) (x) \leq \sum_{k=1}^{\infty} |\lambda_k| S_1 \left(\sqrt{L_w}(\beta_k) \right) (x).$$

By this and (4.6), we know that

$$\begin{aligned} \left\| S_1 \left(\sqrt{L_w}(f) \right) \right\|_{L^p(w, \mathbb{R}^n)} &\leq \left[\sum_{k=1}^{\infty} \int_{\mathbb{R}^n} |\lambda_k|^p \left[S_1 \left(\sqrt{L_w}(\beta_k) \right) (x) \right]^p w(x) dx \right]^{1/p} \\ &\lesssim \left[\sum_{k=1}^{\infty} |\lambda_k|^p \right]^{1/p} \lesssim \|\nabla f\|_{H_w^p(\mathbb{R}^n)}. \end{aligned}$$

Thus, (4.6) implies (4.4).

It remains to prove (4.6). Let a be an $H_w^{1,p}(\mathbb{R}^n)$ -atom associated to a ball $B := (x_B, r_B)$ with $x_B \in \mathbb{R}^n$ and $r_B \in (0, \infty)$. Then, by the Hölder inequality, we find that

$$\begin{aligned} (4.8) \quad &\left\| S_1 \left(\sqrt{L_w}(a) \right) \right\|_{L^p(w, \mathbb{R}^n)}^p \\ &= \int_{\mathbb{R}^n} \left[S_1 \left(\sqrt{L_w}(a) \right) (x) \right]^p w(x) dx \\ &= \sum_{j=0}^{\infty} \int_{U_j(B)} \left[\iint_{\Gamma(x)} \left| t L_w e^{-t^2 L_w}(a)(y) \right|^2 w(y) \frac{dy}{w(B(x, t))} \frac{dt}{t} \right]^{\frac{p}{2}} w(x) dx \\ &\leq \sum_{j=0}^{\infty} [w(2^j B)]^{1-\frac{p}{2}} \left[\int_{U_j(B)} \iint_{\Gamma(x)} \left| t L_w e^{-t^2 L_w}(a)(y) \right|^2 \frac{w(y) dy}{w(B(x, t))} \frac{dt}{t} w(x) dx \right]^{\frac{p}{2}} \\ &\lesssim \sum_{j=3}^{\infty} [w(2^j B)]^{1-\frac{p}{2}} \left[\iint_{R(U_j(B))} \left| t L_w e^{-t^2 L_w}(a)(y) \right|^2 w(y) dy \frac{dt}{t} \right]^{\frac{p}{2}} \\ &\quad + [w(B)]^{1-\frac{p}{2}} \left\| S_1 \left(\sqrt{L_w}(a) \right) \right\|_{L^2(w, \mathbb{R}^n)}^p \\ &\lesssim \sum_{j=3}^{\infty} [w(2^j B)]^{1-\frac{p}{2}} \left[\int_0^{\infty} \int_{(2^{j-2} B)^c} \left| t^2 L_w e^{-t^2 L_w}(a)(y) \right|^2 w(y) dy \frac{dt}{t^3} \right]^{\frac{p}{2}} \\ &\quad + \sum_{j=3}^{\infty} [w(2^j B)]^{1-\frac{p}{2}} \left[\int_{2^{j-2} r_B}^{\infty} \int_{2^{j-2} B} \left| t^2 L_w e^{-t^2 L_w}(a)(y) \right|^2 w(y) dy \frac{dt}{t^3} \right]^{\frac{p}{2}} \\ &\quad + [w(B)]^{1-\frac{p}{2}} \left\| S_1 \left(\sqrt{L_w}(a) \right) \right\|_{L^2(w, \mathbb{R}^n)}^p =: \text{I} + \text{II} + \text{III}, \end{aligned}$$

where $U_j(B)$ is as in (1.7) and $R(U_j(B))$ is as in (1.6) with F replaced by $U_j(B)$.

From (4.7) and the fact that a is an $H_w^{1,p}(\mathbb{R}^n)$ atom, it follows that

$$(4.9) \quad \text{III} \lesssim [w(B)]^{1-\frac{p}{2}} \|\nabla a\|_{L^2(w, \mathbb{R}^n)}^p \lesssim 1.$$

For II, by the assumption that $\{t L_w e^{-t^2 L_w}\}_{t \geq 0}$ satisfies $L^r - L^2$ weighted full off-diagonal estimates with $r \in (1, 2)$, $w \in A_q(\mathbb{R}^n)$ with $q \in [1, \frac{2p}{2-p}(\frac{1}{r} - \frac{1}{2} + \frac{1}{n})]$, and Lemma 2.2, we conclude that

$$(4.10) \quad \text{II} = \sum_{j=3}^{\infty} [w(2^j B)]^{1-\frac{p}{2}} \left[\int_{2^{j-2} r_B}^{\infty} \int_{2^{j-2} B} \left| t^2 L_w e^{-t^2 L_w}(a)(y) \right|^2 w(y) dy \frac{dt}{t^3} \right]^{\frac{p}{2}}$$

$$\begin{aligned}
&\lesssim \sum_{j=3}^{\infty} [w(2^j B)]^{1-\frac{p}{2}} \left\{ \int_{2^{j-2}r_B}^{\infty} t^{-2n(\frac{1}{r}-\frac{1}{2})} \left[\int_B |a(y)|^r [w(y)]^{\frac{r}{2}} dy \right]^{\frac{2}{r}} \frac{dt}{t^3} \right\}^{\frac{p}{2}} \\
&\lesssim \sum_{j=3}^{\infty} [2^{qnj} w(B)]^{1-\frac{p}{2}} (2^j r_B)^{-p(1+\frac{n}{r}-\frac{n}{2})} \left[\int_B |a(y)|^r [w(y)]^{\frac{r}{2}} dy \right]^{\frac{p}{r}}.
\end{aligned}$$

From the Hölder inequality, the fact that a is an $H_w^{1,p}(\mathbb{R}^n)$ -atom and the weighted Sobolev inequality (2.17) with $p_+ = \frac{2n}{n-2}$, it follows that

$$\begin{aligned}
(4.11) \quad &\left[\int_B |a(y)|^r [w(y)]^{\frac{r}{2}} dy \right]^{\frac{p}{r}} \\
&\leq \left[\int_B |a(y)|^{p_+} [w(y)]^{\frac{p_+}{2}} dy \right]^{\frac{p}{p_+}} |B|^{p(\frac{1}{r}-\frac{1}{p_+})} \\
&\lesssim \left[\int_B |\nabla a(y)|^2 w(y) dy \right]^{\frac{p}{2}} |B|^{p(\frac{1}{r}-\frac{1}{2}+\frac{1}{n})} \lesssim [w(B)]^{\frac{p}{2}-1} (r_B)^{p(\frac{n}{r}-\frac{n}{2}+1)}.
\end{aligned}$$

Combining this, (4.10) and the fact that $1 \leq q < \frac{2p}{2-p}(\frac{1}{r} - \frac{1}{2} + \frac{1}{n})$, we conclude that

$$(4.12) \quad \Pi \lesssim \sum_{j=3}^{\infty} 2^{qnj(1-\frac{p}{2})} 2^{-jp(\frac{n}{r}-\frac{n}{2}+1)} \lesssim 1.$$

For I, we write

$$\begin{aligned}
(4.13) \quad \text{I} &\leq \sum_{j=3}^{\infty} [w(2^j B)]^{1-\frac{p}{2}} \left[\int_0^{2^j r_B} \int_{(2^{j-2}B)^c} \left| t^2 L_w e^{-t^2 L_w}(a)(y) \right|^2 w(y) dy \frac{dt}{t^3} \right]^{\frac{p}{2}} \\
&\quad + \sum_{j=3}^{\infty} [w(2^j B)]^{1-\frac{p}{2}} \left[\int_{2^j r_B}^{\infty} \int_{(2^{j-2}B)^c} \cdots \right]^{\frac{p}{2}} =: \text{I}_1 + \text{I}_2.
\end{aligned}$$

By Proposition 2.6, the fact that $w \in A_q(\mathbb{R}^n)$ with $1 \leq q < \frac{2p}{2-p}(\frac{1}{r} - \frac{1}{2} + \frac{1}{n})$, Lemma 2.2 and (4.11), we find that

$$\begin{aligned}
(4.14) \quad \text{I}_1 &\lesssim \sum_{j=3}^{\infty} [w(2^j B)]^{1-\frac{p}{2}} \left[\int_0^{2^j r_B} t^{-2n(\frac{1}{r}-\frac{1}{2})} e^{-\frac{(2^j r_B)^2}{ct^2}} \|a\|_{L^r(w^{\frac{r}{2}}, B)}^2 \frac{dt}{t^3} \right]^{\frac{p}{2}} \\
&\lesssim \sum_{j=3}^{\infty} [w(2^j B)]^{1-\frac{p}{2}} \|a\|_{L^r(w^{\frac{r}{2}}, B)}^p \\
&\quad \times \left[\int_0^{2^j r_B} \left(\frac{2^j r_B}{t} \right)^{2n(\frac{1}{r}-\frac{1}{2}+\frac{3}{2n})} e^{-\frac{(2^j r_B)^2}{ct^2}} \left(\frac{1}{2^j r_B} \right)^{2n(\frac{1}{r}-\frac{1}{2}+\frac{3}{2n})} dt \right]^{\frac{p}{2}} \\
&\lesssim \sum_{j=3}^{\infty} [2^{qnj} w(B)]^{1-\frac{p}{2}} (2^j r_B)^{-p(\frac{n}{r}-\frac{n}{2}+1)} [w(B)]^{\frac{p}{2}-1} (r_B)^{p(\frac{n}{r}-\frac{n}{2}+1)}
\end{aligned}$$

$$\lesssim \sum_{j=3}^{\infty} 2^{qnj(1-\frac{p}{2})} 2^{-jp(\frac{n}{r}-\frac{n}{2}+1)} \lesssim 1.$$

From an argument similar to that used in the above, it also follows that

$$(4.15) \quad I_2 \lesssim 1.$$

Combining (4.15), (4.14), (4.13), (4.12), (4.9) and (4.8), we obtain (4.6). This further implies (4.3). Therefore, we have

$$(4.16) \quad \left(H_{L_w, \text{Riesz}}^p(\mathbb{R}^n) \cap L^2(w, \mathbb{R}^n) \right) \subset \left(H_{L_w}^p(\mathbb{R}^n) \cap L^2(w, \mathbb{R}^n) \right).$$

From (4.16) and (4.2), we deduce that

$$\left(H_{L_w, \text{Riesz}}^p(\mathbb{R}^n) \cap L^2(w, \mathbb{R}^n) \right) = \left(H_{L_w}^p(\mathbb{R}^n) \cap L^2(w, \mathbb{R}^n) \right).$$

This, together with (4.1) and (4.3), implies that $H_{L_w, \text{Riesz}}^p(\mathbb{R}^n) \cap L^2(w, \mathbb{R}^n)$ and $H_{L_w}^p(\mathbb{R}^n) \cap L^2(w, \mathbb{R}^n)$ coincide with equivalent quasi-norms. Then, by a density argument, we complete the proof of Theorem 1.4. \square

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